Effects of High Order Sliding Mode Guidance and Observers On Hit-to-Kill Interceptions

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Maneuvering targets present challenges to missile design in the areas of sensor, tracker, flight control, lethality systems, missile systems integration, and in the design of guidance, navigation, and control algorithms in particular. Classical Proportional Navigation (PN) and Augmented PN (APN) work against a wide range of targets including some maneuvering targets. However for robustness these algorithms require the interceptor to have a maneuver advantage over the target of at least 3:1. The effects of smooth second order sliding mode guidance laws, and second order sliding mode observers/differentiators on hit-to-kill interceptions are studied in this paper. It also explores the effects of using sliding mode guidance with Kalman filtering, sliding mode guidance with sliding mode observers/differentiators, and classical PN with sliding mode observers/differentiators. Sliding mode observers/differentiators dramatically reduce the complexity of target state estimation. The algorithms presented here have been verified using the high fidelity Phoenix six-degree-of-freedom (6-DOF) developed at Radiance Technologies.

I. Introduction

Two important measures of guidance system performance are: 1) target-to-interceptor maneuver ratio (or maneuver ratio) and 2) miss distance, the ultimate measure. The performance of interceptor subsystems such as the seeker, radome, flight control system, airframe, guidance computer, their associated lags, uncertainties, and interceptor flight regime all impact the miss performance of the interceptor. The guidance law, overall response time of the interceptor guidance and control loops, and the type of target maneuver have the most impact the maneuver ratio. Miss performance optimization requires the classic trade between bandwidth and smoothness. Although the trades to get to the specific parameter values of the filters and observers will not be presented in the...
paper, it is however a significant part of the design and analysis process, particularly in a non-linear 6-DOF simulation environment.

This paper examines a low altitude maneuvering Theater Ballistic Missile (TBM) scenario. Table 1 shows the nominal conditions for this scenario, key interceptor parameters, and key target parameters.

<table>
<thead>
<tr>
<th></th>
<th>Launch</th>
<th>Interceptor Intercept Conditions</th>
<th>Target Intercept Conditions</th>
<th>Target Maneuver</th>
<th>Homing Time</th>
<th>Update Rates</th>
<th>Continuous Elements</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Sea Level</td>
<td>Altitude 6km</td>
<td>Speed 1.9 km/s</td>
<td>Step</td>
<td>Approximately 4sec</td>
<td>Seeker 100Hz</td>
<td>1kHz</td>
</tr>
<tr>
<td></td>
<td>45 deg Elevation</td>
<td>Speed 1.0 km/s</td>
<td>-45deg Elevation 180 deg Azimuth</td>
<td>4 Sec. Time Constant</td>
<td>Acquisition Range 13km</td>
<td>Noise 0.1mr, 1σ Range 5m, 1σ</td>
<td>4th Order Runge Kutta Integration</td>
</tr>
<tr>
<td></td>
<td>0 deg Azimuth</td>
<td></td>
<td></td>
<td></td>
<td>Point Seeker toward Expected Target 14.5km</td>
<td>Filters/Observers 100Hz</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Guidance 100Hz</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Autopilot 2nd Order Pole Placement 100Hz</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Navigation 100Hz</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Parameter Estimation 100Hz</td>
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</tbody>
</table>

The main goal of this paper is to evaluate miss performance and maneuver ratio of the aforementioned combinations of filters, observers and guidance laws in a stochastic 6-DOF simulation. The smooth High (Second) Order Sliding Mode (SOSM) guidance law\(^4,5\) and observers/differentiators\(^6,7\) used for the missile-target engagement states estimation are studied.

From guidance and tracking perspective the most important advantage the 6-DOF simulation has over lower fidelity simulations is detailed modeling of the Line-of-Sight (LOS) reconstruction process. Although often simplified in 3-DOF and linear simulations, LOS reconstruction details, particularly in the presence of boresight slope error, should be included in detailed guidance design and analysis. If warranted, errors due to radome or seeker window aberration can be treated by real-time on-board methods\(^8\). A generic 2nd order pole placement autopilot has been used here in place of the SOSM-based autopilot\(^5,9\). The SOSM-based autopilot\(^5,9\), or one of similar form, will be integrated into the system in future studies. Recent progress in high order sliding mode guidance and control\(^4,5,10-16\), both numerically and theoretically, indicates that this more detailed examination of sliding mode guidance and observers is warranted.

This paper is structured as follows. Section II presents the Smooth Second Order Sliding Mode Control. The Robust to Noise Second Order Sliding Mode Observers/Differentiators are discussed in Section III. Section IV presents Guidance Algorithm Implementations. Computer Simulations and Performance Analysis are discussed in Section V. Finally, Section VI presents the conclusions.

### II. Smooth Second Order Sliding Mode Control

Consider a SISO sliding variable dynamics given as

\[
\sigma = f(\sigma, t) + u, \tag{1}
\]

where \(\sigma \in \mathbb{R}^1\) is the sliding variable, such that \(\sigma = 0\) defines the system motion on the sliding surface, \(u \in \mathbb{R}^1\) is a control input that is supposed to be smooth, and \(f(\sigma, t)\) is an uncertain smooth nonlinear time-varying function \(|f(\sigma, t)| \leq L, L > 0, \forall |\sigma| \leq \bar{\sigma}|\).

The proposed control algorithm that smoothly stabilizes \(\sigma\) at zero is formulated in the following Theorem. *Theorem*\(^4,16\). Let \(\alpha_0 > 0, \alpha_1 > 0\) and uncertain term \(f(\sigma, t) = 0\) in (1) then the smooth control

\[
u = -\alpha_0 |\sigma|^{1/2} \text{sign}(\sigma) - \alpha_1 |\sigma|^{1/3} \text{sign}(\sigma)d\tau \tag{2}
\]

provides for the finite time convergence of the sliding surface compensated dynamics (1) into 2-sliding mode \(\sigma = 0\).
Proof. First of all, an asymptotic convergence will be proved. The system (1) and (2) can be equivalently presented by the system of two first-order equations

\[
\begin{align*}
\dot{x}_1 &= x_2 - \alpha_1 |x_1|^{1/2} \text{sign}(x_1) \\
\dot{x}_2 &= -\alpha_0 |x_1|^{1/3} \text{sign}(x_1)
\end{align*}
\]  

(3)

where \( x_1 = \sigma \), and \( x_2 = -\alpha_0 \int_0^{\sigma} \left| \sigma \right|^{1/3} \text{sign}(\sigma) \, d\tau \).

Let a Lyapunov function candidate be

\[
V(x_1, x_2) = \frac{x_1^2}{2} + \int_0^{x_1} \alpha_0 \left| \sigma \right|^{1/3} \text{sign}(\sigma) \, d\sigma ,
\]

(4)

\( V(x) > 0 \), if \( x \in \mathbb{R}^2 \setminus \{0\} \), then the Liapunov function derivative will be

\[
\dot{V} = \frac{\partial V}{\partial x} \left[ x_2 - \alpha_1 |x_1|^{1/2} \text{sign}(x_1) \right] + \frac{\partial V}{\partial x_2} \left[ -\alpha_0 |x_1|^{1/3} \text{sign}(x_1) \right] < 0, \quad x \in \mathbb{R}^2 \setminus \{0\}
\]

(5)

Next, applying La Salle theorem\(^7\) we can prove \( x \to 0 \) as time increases. A set \( x : \left[ \dot{V}(x) = 0 \right] \) consists of \( x_1 = 0 \) and \( x_2 \) equal to any real value. Substituting these values into (3) we obtain \( \dot{x}_1 = x_2, \dot{x}_2 = 0 \). It means that \( x_2 = c_1 \) is constant and \( x_1 = c_2 + c_1 t \). However, we know that in the equilibrium point \( x_1 = 0 \). So, \( c_1 = c_2 = 0 \), and \( x_2 = 0 \) in the equilibrium. So, we proved asymptotic convergence \( x_1 \) and \( x_2 \) to zero. In the other words \( \sigma \to 0 \) and \( \dot{\sigma} \to 0 \) as time increases, and we have smooth asymptotic 2nd order sliding that is achieved without a disturbance term.

Second of all, finite time convergence will be proved using the result formulated into the following Proposition\(^8\).

Proposition. Let \( x \in D \subset \mathbb{R}^n, \dot{x} = f(x), f : \mathbb{R}^n \to \mathbb{R}^n \) is continuous on an open neighborhood \( D \) of the origin and locally Lipshitz on \( D \setminus \{0\} \) and \( f(0) = 0 \). Suppose there is a continuous function \( V : D \to \mathbb{R} \) such that the following conditions hold

(i) \( V \) is positive definite;
(ii) \( \dot{V} \) is negative on \( D \setminus \{0\} \);
(iii) there exist real numbers \( k > 0 \) and \( \alpha \in (0, 1) \), and a neighborhood \( N \subset D \) of the origin such that

\[
\dot{V} + k V^{\alpha} \leq 0 \text{ on } N \setminus \{0\} .
\]

Then, the origin is a finite-time-stable equilibrium of \( \dot{x} = f(x) \).

Eqs. (4) and (5) give us the function satisfying (i), (ii) of the theorem. To prove (iii) we consider the validity of

\[
\dot{V} + k V^{\alpha} = -\alpha_1 \alpha_2 |x_1|^{1/2} \text{sign}(x_1) + k \left( \frac{x_1^2}{2} + \frac{1}{2} \alpha_0 |x_1|^{1/3} \right)^\alpha \leq 0
\]

(6)

Inequality (11) can be transformed to

\[
k \left( \frac{x_1^2}{2} + \frac{1}{2} \alpha_0 |x_1|^{1/3} \right) \leq (\alpha_1 \alpha_2)^\alpha |x_1|^{1/2}
\]

(7)

In a small neighborhood of the origin, where \( x_2 \ll 1, x_1 \ll 1 \), and where \( |x_2| \leq \alpha_1 |x_1|^{1/2} \), the left part in (7), will be dominated by \( \frac{x_1^2}{2} + \frac{1}{2} \alpha_0 |x_1|^{1/3} \), which in turn will be dominated by \( |x_1| \), while the right part will be proportional to \( |x_1|^{1/2} \). There exists \( \alpha \in (0, 1) \) such that \( \frac{1}{6} \alpha \leq 1 \), hence one can always select \( k > 0 \) and a neighborhood \( N \subset D \) of the origin such that \( (\alpha_1 \alpha_2)^\alpha |x_1|^{1/2} \) will be dominated over \( |x_1| \) and over the right part in (7), which is proportional to \( |x_1| \) in a small \( N \subset D \). Thus, the proposition (iii) of the theorem is true in the area where \( |x_2| \leq \alpha_1 |x_1|^{1/2} \) and in the neighborhood \( N \subset D \) of the origin such that \((\alpha_0 \alpha_1)^\alpha |x_1|^{1/2} \) will be dominated over \( |x_1| \) and over the right part in (7).
In Figure 1 the flow of the system (3) in the state-coordinates \((x_1, x_2)\) is considered. It is easy to show that the vector field of the system (3) is continuous and locally Lipshitz everywhere on \(\mathbb{R}^2\) except at the origin. Therefore, the flow of the solutions will always escape the areas II, III, V, VI after a finite amount of time, while the areas I and IV contain a small neighborhood \(N \subset D\) of the origin where condition (11) is satisfied; i.e., solutions converge in a finite time. Since global asymptotic stability of the origin has been proven, any state trajectory inevitably enters neighborhood \(N \subset D\) of the origin either in the area I or IV from where it will reach the origin in a finite time.

![Figure 1. The flow of the system (3) in the state-coordinates \((x_1, x_2)\)](image)

Thus, the system (3) has a finite-time stable equilibrium at the origin for any \(\alpha_1 > 0, \alpha_0 > 0\).

The sliding surface compensated dynamics under the proposed control law (2) is sensitive to the unknown bounded term \(f(\sigma, t)\). In order to compensate for this term we propose to use the control law (7) that includes a smooth disturbance estimator \(\hat{f}_{\text{smooth}}(\sigma, t)\) based on the super-twisting algorithm\(^{14,15}\) and a nonlinear proportional-integral type term

\[
u = -\alpha_1 \frac{\sigma}{|\sigma|^{1/2}} - \alpha_0 \left|\frac{\sigma}{|\sigma|^{1/2}}\right|^{1/3} \text{sign}(|\sigma|)dt + \hat{f}_{\text{smooth}}(\sigma, t)\tag{8}
\]

where

\[
\dot{\sigma} = \rho_1 \frac{\sigma - \hat{\sigma}}{|\sigma - \hat{\sigma}|^{1/2}} + \rho_3 \text{sign}(\sigma - \hat{\sigma})dt + \int (\dot{\sigma} + u)dt\tag{9}
\]

and

\[
\hat{f}_{\text{smooth}} = \text{LowPassFilter}(\hat{f})\tag{10}
\]

Remark. The super-twist based observer (9) provides for a finite time convergence, i.e. \(\dot{\sigma} = f\) for all time larger than some finite time instant. On the other hand, \(\dot{\sigma}\) is not smooth, and its use in the control law (11) makes the control law continuous only. In order to retain smoothness of the control law (8) a smooth Low Pass Filter (10) should be used. The Low-Pass Filter (10) is to be implemented as a second-order observer with finite time convergence (for instance, using another super-twisting algorithm in a cascade). Then, provided sufficiently smooth uncertainty \(f(\sigma, t)\), the estimate \(\dot{\sigma}\) in (9), low-pass filtered before entering control as \(\hat{f}_{\text{smooth}}\), converges to \(f(\sigma, t)\) in a finite time. Considering \(\hat{f}_{\text{smooth}} \approx f(\sigma)\), when \(\sigma \ll 1\) and the bandwidth of \(f(\sigma, t)\) is far to the right from the low-pass filter cut-off frequency, in the vicinity of the origin we have the closed-loop asymptotic 2\textsuperscript{nd} order dynamics.
(8). Under the discrete control law \( u(t) = u[kT] = \text{const} \), \( kT \leq t < (k+1)T \), the actual convergence is practically achievable to the domain \( |\sigma| \sim T^2 \).

III. Robust to Noise Second Order Sliding Mode Observers/Differentiators

Developed recently\(^{14}\), an exact differentiator, based on the second order SMC method, can reconstruct a given signal \( f(t) \) and its derivative \( \dot{f}(t) \) provided certain constraint on the signal second derivative \( |\ddot{f}(t)| \leq L \), for known \( L > 0 \). It has the following structure

\[
\begin{align*}
\dot{x}_1 &= \int \dot{x}_2 \, d\tau, \\
\dot{x}_2 &= \rho_1 [f(t) - \dot{x}_1] + \rho_2 \int \text{sgn}(f(t) - \dot{x}_1) \, d\tau
\end{align*}
\]

If we select \( \rho_0 = 4L, \rho_1 = 0.5L^{0.5} \), then \( \dot{x}_1 \rightarrow f(t), \dot{x}_2 \rightarrow \dot{f}(t) \) in a finite time, \( T \), independent of \( \rho_0, \rho_1 \), as proved in\(^{14}\). Since the structure (11) uses the signal state-model

\[
\dot{x}_1 = x_2, \quad \dot{x}_2 = \dot{f}(t), \quad |\dot{f}(t)| \leq L
\]

to estimate the states \( x_1, x_2 \) of a signal, it can be called the state-observer/differentiator.

In case of violation of the condition \( |\dot{f}(t)| \leq L \), when, say \( f(t) = f_o(t) + v(t) \), \( |\dot{f}_o(t)| \leq L, |v(t)| \gg L \), the observer/differentiator (11), which structure is presented in Figure 2, reconstructs the actual signal \( f_o(t) \) with good accuracy: \( \dot{x}_1 \rightarrow f_o(t) \); however, \( \dot{x}_2(t) \) will have significant high-frequency component due to noise, \( v(t) \), amplification on the first “high-gain” term in the expression for \( \dot{x}_2(t) \) (Eq.(11)). It is shown\(^{14}\) that the estimation errors \( e_1 = \dot{x}_1 - f(t) \) and \( e_2 = \dot{x}_2 - \dot{f}(t) \) obtained by the estimator (11) is proportional to \( \varepsilon^{1/2} \), where \( |v(t)| \leq \varepsilon \).

The SOSM-based observation algorithm presented in (11) is known as the Super Twisting Algorithm\(^{14}\).

Figure 2: Super Twist based SOSM Observer/Differentiator

Further improvement of this approach is presented in\(^{6,7}\), where a low-pass nonlinear filter was incorporated in the observer structure producing a third order system instead of that of the forth order in Figure 2. The robust to measurement noise SOSM observer\(^{6,7}\) is given by

\[
\begin{align*}
\dot{x} &= \rho_o \, \text{sgn}(J_o), \\
\dot{x} &= b \frac{e}{|e|^{\beta_3}} - a \frac{(\chi + e)}{|\chi + e|^{\beta_3}}, \\
J_o &= \chi + e,
\end{align*}
\]

The nonlinear system

\[
\dot{x} = b \frac{e}{|e|^{\beta_3}} - a \frac{(\chi + e)}{|\chi + e|^{\beta_3}}, \quad J_o = \chi + e
\]
is called the Nonlinear Dynamic Sliding Manifold (NDSM), where the sliding mode (condition $J_o = 0$) will guarantee the estimation error dynamics

$$\dot{e} = -b|e|^{0.5} \text{sgn}(e),$$

with a finite time convergence to the origin $e = \dot{e} = 0$.

One can notice from Eq. (13) that, when $J_o \to 0$, we have approximately

$$\ddot{x} = \rho_o \text{sgn}(J_o) \Rightarrow J_o \approx \dot{e} + b \frac{e}{|e|^{0.5}},$$

and the observer (13) takes the form

$$\ddot{x} = \rho_o \text{sgn} \left( \dot{e} + b \frac{e}{|e|^{0.5}} \right),$$

which has been presented in the work\textsuperscript{15} as an example of the second-order sliding mode system. The observer (13) requires the information about $\dot{e}$, while the observer (13) with a NDSM doesn’t. Thus, the nonlinear dynamic sliding manifold (4) represents a nonlinear filter/differentiator incorporated into a second-order sliding mode observer. The observer (3) is a third-order system, and so is a three-state Kalman filter, if we consider Kalman gains to be precomputed in advance. The structure of the observer (13) is presented in Figure.3.

![Figure 3: SOSM Estimator based on Nonlinear Dynamic Sliding Manifold](image)

The unique feature of this differentiator, namely the low-pass band property based on the known value for the signal second time-derivative, clearly reveals itself on a very sharp frequency response characteristic. Obtained using the approximate analyzer, the frequency response plots for the observer (3) as a differentiator are presented in Figure 4, which indicate that signals in the range 0-10 rad/sec are differentiated almost exactly (phase lead 90deg., magnitude +20dB/dec), and above that range the attenuation –20dB/dec is provided.

![Figure 4. (a): SMC Differentiator Amplitude [dB] Response ($b = 0.1, a = 10, \rho_o = 50$)](image)  
(b): SMC Differentiator Phase Response ($b = 0.1, a = 10, \rho_o = 50$)
IV. Simulation Set Up and Guidance Algorithm Implementations

A. Simulation set up

Figure 5 provides a functional block diagram of the Phoenix 6-DOF simulation.

[Diagram not shown in text]

The simulation is structured to model various types of filters, observers, guidance laws, autopilots, control effectors, and seekers. For example, in this study a two-degree-of-freedom stabilized seeker was used as a means of following the target and providing seeker measurements. For purposes on LOS reconstruction the first valid seeker LOS measurement is used to define an inertial LOS reference frame, i.e., all filtering and observation functions are performed in this frame. The outputs of the filters and observers are then transformed to the appropriate frame(s) for guidance and control.

B. Sliding mode observers/differentiators

The robust to noise SOSM Differentiator (13) is used to estimate range rate, based on the seeker range measurement. The SOSM range differentiator is formulated in the simulation using the seeker range measurement as its input.

\[
\begin{align*}
\dot{\hat{R}} &= \rho_e \text{sgn}(J_o), \\
\dot{\hat{\chi}} &= b \left( e - a \frac{(\chi + e)}{\sqrt{|\chi + e|^2}} \right), \\
J_o &= \chi + e,
\end{align*}
\]

where

\[
R_M = \text{Seeker Range measurement, } \hat{R} = \text{Estimated Range, } e = \text{Observation Error, } \hat{\chi} = \text{Estimated Range Acceleration, } \dot{\hat{R}} = \text{Estimated Range Rate or Closing Velocity. The filter parameters } a \text{ and } b \text{ are set, } b = 0.1, a = 10, \rho_e = 50. \text{ These are the same values that were presented in the differentiator example above.}
\]

Also, the twisting estimation algorithm\(^{15}\) is used for LOS rate estimation has the form shown below:

\[
\begin{align*}
u_1 &= K_1 \lambda_M, \\
u_2 &= K_2 \left( \omega_M + \frac{dL}{R} \Delta t \right) \\
\dot{\lambda}_M &= \frac{\hat{x}}{K_1}, \omega_M = \frac{\hat{x}}{K_2}
\end{align*}
\]

where:

[Equation not shown in text]
\( \hat{\lambda}_M \) = Measured LOS Angle, \( \hat{\omega}_M \) = Measured LOS Rate, \( \hat{\lambda} \) = Estimated LOS Angle, \( \hat{\omega} \) = Estimated LOS Rate, \( a_L \) = Interceptor Acceleration Normal to LOS, \( \hat{R} \) = Estimated Range, \( \Delta t \) = Observer Update Period

The inputs to the observer are the measured LOS angle and the rate command to the seeker platform. This command is effectively a measurement of the LOS rate. The observer outputs estimated LOS angle and rate, respectively. The acceleration normal to the LOS helps the observer to converge more quickly to a good estimate of the LOS rate without having to increase the gains.

C. Kalman Filter

The Kalman filtering\textsuperscript{1,19} approach uses three decoupled two-state Kalman filters. The pitch and yaw filters estimate the LOS angle and rate in both axes, based on reconstructed measurements of the LOS angle in both axes. The range axis filter estimates range and rate based on the measurement of the seeker range.

\[
\begin{align*}
\text{Pitch} \\
\begin{bmatrix}
\hat{\lambda}_p(n) \\
\hat{\omega}_p(n)
\end{bmatrix} &= \begin{bmatrix} 1 & \Delta t \\
0 & 1 + \frac{\Delta t}{2} G \end{bmatrix} \begin{bmatrix}
\hat{\lambda}_p(n-1) \\
\hat{\omega}_p(n-1)
\end{bmatrix} + \begin{bmatrix} 0 \\
\frac{\Delta t}{R_{MEB}}
\end{bmatrix} a_{zL} \\
\text{Yaw} \\
\begin{bmatrix}
\hat{\lambda}_y(n) \\
\hat{\omega}_y(n)
\end{bmatrix} &= \begin{bmatrix} 1 & \Delta t \\
0 & 1 + \frac{\Delta t}{2} G \end{bmatrix} \begin{bmatrix}
\hat{\lambda}_y(n-1) \\
\hat{\omega}_y(n-1)
\end{bmatrix} + \begin{bmatrix} 0 \\
\frac{\Delta t}{R_{MEB}}
\end{bmatrix} a_{zL} \\
\text{Range} \\
\begin{bmatrix}
R(n) \\
V(n)
\end{bmatrix} &= \begin{bmatrix} 1 & \Delta t \\
0 & 1 + \frac{\Delta t}{2} G \end{bmatrix} \begin{bmatrix}
R(n-1) \\
V(n-1)
\end{bmatrix} + \begin{bmatrix} 0 \\
\frac{\Delta t}{R_{MEB}}
\end{bmatrix} a_{zL}
\end{align*}
\]

where:
\( \hat{\lambda}_p \) = Pitch Estimated LOS Angle, \( \hat{\lambda}_y \) = Yaw Estimated LOS Angle, \( \hat{\omega}_p \) = Pitch Estimated LOS Rate, \( \hat{\omega}_y \) = Yaw Estimated LOS Rate, \( R \) = Estimated Range, \( V \) = Estimated Closing Velocity, \( a_{zL}, a_{zL}, a_{zL} \) = LOS Frame Interceptor Accelerations.

The sensed interceptor acceleration in the LOS frame is added to the Kalman filter state equations as a known deterministic input. The full set of Kalman filter equations can be found in many references such as Gelb\textsuperscript{19}, and are not presented here.

D. LOS Reconstruction

The initial LOS frame is defined by the first valid LOS measurement. This provides an inertial (or at least non-rotating) frame that is close to the measurement frame that allows use small angle approximations and decoupled filters and observers. If an earth centered inertial (ECI) frame were selected, small angle approximations would no longer apply. The Kalman Filter would then become an Extended Kalman Filter (EKF). The EKF form would add complexity and could have convergence or stability problems under certain conditions. For strapdown seeker applications a synthetic tracker can be used to simplify LOS reconstruction\textsuperscript{20}. The initial LOS frame is constructed by using the following transformation matrices:

\[
C_{LOS}^I = C_P^B C_B^G C_{LOS}^G
\]

where:
\( C_{LOS}^I \) = Initial LOS Frame to Inertial Frame Transformation Matrix
\( C_P^B \) = Body Frame to LOS Frame Transformation Matrix
\( C_B^G \) = Inner Gimbal Frame to Body Frame Transformation Matrix
\( C_{LOS}^G \) = Initial LOS in Seeker Frame to Inner Gimbal Transformation Matrix

All seeker measurements that follow are then processed in the following manner. First a unit vector is created from the seeker measurements range, azimuth, and elevation angles via the following set of equations:

\[
\begin{align*}
u_{XS} &= \cos(\varepsilon_x) \cos(\varepsilon_y) \\
u_{YS} &= \cos(\varepsilon_x) \sin(\varepsilon_y) \\
u_{ZS} &= -\sin(\varepsilon_x)
\end{align*}
\]
The unit vector in (21) is then transformed from the current LOS frame to the inertial frame by

\[
C_{LOS}^I = C_{G} B C_{LOS}^G
\]

\[
\vec{u}_{MI} = [C_{LOS}^I] \vec{u}_{MS}
\]

(22)

where:

- \(C_{LOS}^I\) = Current LOS Frame to Inertial Frame Transformation Matrix
- \(C_{G}^B\) = Inner Gimbal Frame to Body Frame Transformation Matrix
- \(C_{LOS}^G\) = Current LOS in Seeker Frame to Inner Gimbal Transformation Matrix

This unit vector is then transformed to the initial LOS frame via

\[
\vec{u}_{LI} = [C_{LOS}^I] \vec{u}_{IS}
\]

(23)

The LOS angles are then reconstructed

\[
\begin{align*}
\epsilon_{YL} &= A \tan(u_{LY} / u_{LX}) \\
\epsilon_{PL} &= -A \sin(u_{LZ})
\end{align*}
\]

(24)

where:

- \(\epsilon_{YL}\) = Yaw LOS angle in the initial LOS frame
- \(\epsilon_{PL}\) = Pitch LOS angle in the initial LOS frame
- \(u_{LY}\) = X component of the LOS unit vector in the initial LOS frame
- \(u_{LY}\) = Y component of the LOS unit vector in the initial LOS frame
- \(u_{LZ}\) = Z component of the LOS unit vector in the initial LOS frame

The same process is used to construct the LOS rate commands in the initial LOS frame. The LOS rate commands are used in the super twist algorithm but are not needed by the Kalman filter. The Gimbal transformation matrices are computed from the gimbal feedbacks; the body-to-inertial transformation matrix is computed by the navigation system based on the Inertial Measurement Unit (IMU) outputs.

E. Guidance Laws

The guidance laws analyzed and presented in this paper are PN, APN, and Second Order Sliding Mode.

1. Proportional Navigation (PN) and Augmented Proportional Navigation (APN) guidance laws.

Zarchan\(^1\) presents a readable treatment of PN and APN. PN is implemented the Phoenix 6-DOF simulation as

\[
\eta_{CP} = \frac{K_G V_C \hat{\omega}_{JP}}{X_{LOS}^B}
\]

\[
\eta_{CY} = \frac{K_G V_C \hat{\omega}_{JY}}{X_{LOS}^B}
\]

(25)

where:

- \(\eta_{CP}\) = Pitch Acceleration Command, \(\eta_{CY}\) = Yaw Acceleration Command
- \(K_G\) = Guidance Gain, \(V_C\) = Estimated Target-to-Missile Closing Velocity
- \(\hat{\omega}_{JP}\) = Estimated Pitch LOS Rate, \(\hat{\omega}_{JY}\) = Estimated Yaw LOS Rate
- \(X_{LOS}^B\) = X component of the LOS unit vector in body frame (approximates cosine of the angle between the LOS and the body or interceptor velocity, assuming small AoA)

APN is implemented in the Zero Effort Miss (ZEM) form. Zarchan\(^1\) proves that this is equivalent to the usual polar form. For completeness the polar form is shown here as

\[
\eta_C = K_G \left( V_C \hat{\omega}_h + \frac{1}{2} A_{TN} \right)
\]

(26)

The equivalent Zero Effort Miss form of APN is implemented in the 6-DOF simulation in the following manner. The zero effort miss vector in the inertial frame is
\[ \tilde{M}_E = \Delta \tilde{R} + \Delta \tilde{V}_{tGO} + \frac{1}{2} \Delta \tilde{A}_{tGO}. \]  

(27)

The zero effort miss vector is rotated into the body frame as

\[ \tilde{M}_{EB} = [C] B \tilde{M}_E \]  

(28)

The guidance commands are constructed as

\[ \eta_{CP} = \frac{K_G M_{ZB}}{t_{GO}^2 X_{BLOS}} \]
\[ \eta_{CY} = \frac{K_G M_{YB}}{t_{GO}^2 X_{BLOS}} \]  

(29)

where:

- \( \eta_{CP} \) = Pitch Acceleration Command,
- \( \eta_{CY} \) = Yaw Acceleration Command,
- \( K_G \) = Guidance Gain,
- \( X_{BLOS} \) = X component of the LOS unit vector in body frame (approximates cosine of the angle between the LOS and the body, or interceptor velocity, assuming small AoA),
- \( t_{GO} \) = Estimated Time-to-go until Intercept,
- \( M_{ZB} \) = Z Component of Estimated Miss in the Interceptor Body Frame
- \( M_{YB} \) = Y Component of Estimated Miss in the Interceptor Body Frame

2. Smooth Second Order Sliding Mode (SOSM) guidance law.

A theoretical development of SOSM guidance is based on smooth SOSM control presented in Section II. In polar coordinates the relative position is presented by \( \mathbf{R} = (r, \lambda) \), where \( r \) = range along Line-Of-Site (LOS), and \( \lambda \) = LOS angle. The following planar state model of homing-missile engagement kinematics is used\(^4,5\)

\[
\begin{cases}
\dot{r} = V_r, \\
\dot{\lambda} = V_\lambda, \\
\dot{V}_r = \frac{V_r}{r} + A_r - \sin(\lambda - \gamma_m) n_L, \\
\dot{V}_\lambda = \frac{V_\lambda}{r} - \cos(\lambda - \gamma_m) n_L,
\end{cases}
\]  

(30)

where \( V_\lambda = r \omega_\lambda \), and we consider \( \omega_\lambda \) as a commanded output, missile normal acceleration as a control input, and projections of target acceleration along and orthogonal to LOS, \( A_r, A_\lambda \), are considered as unknown bounded disturbances.

It well known that for a direct hit, it’s necessary to keep \( V_r < 0 \). It was shown in\(^4,5,13\) that a direct hit can be achieved if \( \omega_\lambda = 0 \) or \( V_\lambda = 0 \). Another less aggressive hit-to-kill guidance strategy is proven to be\(^4,5,13\)

\[ \omega_\lambda = \frac{c_0}{\sqrt{r}} \quad \text{or} \quad V_\lambda = c_0 \sqrt{r}, \]  

(31)

where \( c_0 \) is some constant.

Now, the following guidance task can be formulated: stabilize the system (1) or (2) on the manifold

\[ \sigma_1 = \omega_\lambda = 0, \quad \text{or} \quad \sigma_1 = V_\lambda = 0 \]  

(32)

or

\[ \sigma_2 = \omega_\lambda - \frac{c_0}{\sqrt{r}} = 0, \quad \text{or} \quad \sigma_2 = V_\lambda - c_0 \sqrt{r} = 0 \]  

(33)

where the quantity \( \sigma_i, i = 1,2 \) determines the system (30) output to stabilize to zero.

The guidance control law is formulated in terms of normal acceleration command \( n_{CL} \) that stabilizes (32) or (33). This command is usually followed by means of corresponding aerodynamic surface deflections that can be treated as a control function (autopilot) compensating for the interceptor dynamics. Unlike in the work\(^5\), where various continuous approximations of SMC were studied for the normal acceleration command, \( n_{CL} \), design, in this
work we use SOSM-based control. The expected advantages are in increasing robustness and accuracy of hit-to-kill intercept.

From (30), (33) the σ-dynamics is identified as (we omit the subscript)
\[
\dot{\sigma} = -\frac{V_r V_\lambda}{r} + A_{T,\lambda} - \frac{c_0 V_r}{2\sqrt{r}} - \cos(\lambda - \gamma_M) n_L,
\]  
so the commanded acceleration, \(n_{Lc}\), for the interceptor normal acceleration \(n_L\) is selected to be
\[
n_{Lc} = \frac{1}{\cos(\lambda - \gamma_M)} \left( \alpha_1 \left| \sigma \right|^{1/2} + \alpha_0 \int \left| \sigma \right|^{1/2} \text{sign}(\sigma) d\tau - N' \frac{V_r V_\lambda}{r} - \frac{c_0 V_r}{2\sqrt{r}} + \hat{A}_{\text{smooth}, T,\lambda} \right), \quad N' = 1
\]
where the target acceleration norm al to LOS is smoothly estimated using Super-Twisting observer\textsuperscript{14,15} with a consecutive low-pass filtering in order to provide for a desired degree of smoothness. This is
\[
\hat{A}_{T,\lambda} = \rho_{\lambda} \left( \frac{\sigma - \hat{\sigma}}{\left| \sigma - \hat{\sigma} \right|^{1/2}} \right) + \rho_{\delta} \int \text{sign}(\sigma - \hat{\sigma}) d\tau, \quad \dot{\sigma} = \left( \hat{A}_{T,\lambda} - \cos(\lambda - \gamma_M) n_{Lc} \right) d\tau,
\]
\[
\hat{A}_{\text{smooth}, T,\lambda} = \text{LowPassFilter}\{\hat{A}_{T,\lambda}\}
\]
The homing (when a seeker is activated and the designed guidance law is executed) closed loop interceptor kinematics (30), (33)-(35) is derived as follows:
\[
\begin{align*}
\dot{r} &= V_r, \\
\dot{V}_r &= c_0^2 + A_{T,\lambda} - \sin(\lambda - \gamma_M) n_L, \\
\dot{\lambda} &= \frac{c_0 V_r}{\sqrt{r}}, \\
\dot{V}_\lambda &= c_0 \frac{V_r}{\sqrt{r}},
\end{align*}
\]  
Since the initial condition \(V_r(0) = M < 0\), it is easy to identify \(c_0\) such that \(V_r(t) < 0 \quad \forall t \leq t_{go}\), where \(t_{go}\) is time required for the interceptor to hit the target. So, the guidance law (15) based on asymptotic smooth second order sliding mode control inevitably guarantees hit-to-kill intercept.

Although, estimating \(A_{T,\lambda}\) via Super-Twisting algorithm (37) is simpler then using Kalman Filter\textsuperscript{16}, in this work we study planar smooth SOSM guidance law (35) without estimating \(A_{T,\lambda}\) that further simplifies the proposed SOSM guidance. This is
\[
\eta_{CL} = \frac{1}{\cos(\lambda - \gamma_M)} \left( \alpha_1 \left| \sigma \right|^{1/2} \text{sign}(\sigma) + \alpha_0 \int \left| \sigma \right|^{1/2} \text{sign}(\sigma) d\tau - N' \frac{V_r V_\lambda}{r} - \frac{c_0 V_r}{2\sqrt{r}} \right)
\]
In this work the SOSM guidance law (38) is implemented with \(N' = 4\) and \(c_0 = 0\), which yields the following planar guidance law:
\[
\eta_{CL} = \frac{1}{\cos(\lambda - \gamma_M)} \left( A_{PN} + A_{SOSM} \right)
\]
where
\[
A_{PN} = -4V_r \hat{\omega}_\lambda, \quad A_{SOSM} = 3 \left| \sigma \right|^{1/2} \text{sign}(\sigma) + \alpha_0 \int \left| \sigma \right|^{1/2} \text{sign}(\sigma) d\tau, \quad \sigma = \hat{\omega}_\lambda \dot{R},
\]
\(\hat{\omega}_\lambda = \text{Estimated LOS Rate, } \dot{R} = \text{Estimate Range, } \lambda = \text{LOS Angle, } \gamma_M = \text{Interceptor flight path angle, } \alpha_0 = \text{Gain Term.}\)

Note: The signs of the acceleration commands will be a function of command plane.

The planar SOSM guidance law (39) is expanded into the 6-DOF simulation as the following sets of equations
\[ \eta_{1P} = K_G \hat{V}_C \hat{\omega}_{JP} + 3|\sigma_{0P}|^{1/2} \text{sign}(\sigma_{0P}) \], \quad \eta_{1Y} = K_G \hat{V}_C \hat{\omega}_{3Y} + 3|\sigma_{0Y}|^{1/2} \text{sign}(\sigma_{0Y}) \\
\hat{\eta}_{2P}(n) = |\sigma_{0P}|^{1/2} \text{sign}(\sigma_{0P}) \], \quad \eta_{2P}(n) = \eta_{2P}(n-1) + \hat{\eta}_{2P}(n) \Delta t \\
\hat{\eta}_{2Y}(n) = |\sigma_{0Y}|^{1/2} \text{sign}(\sigma_{0Y}) \], \quad \eta_{2Y}(n) = \eta_{2Y}(n-1) + \hat{\eta}_{2Y}(n) \Delta t \\
\eta_{CP} = (\eta_{1P} + \alpha_0 \hat{\eta}_{2P}) / \hat{X}_{BLOS}, \quad \eta_{CY} = (\eta_{1Y} + \alpha_0 \hat{\eta}_{2Y}) / \hat{X}_{BLOS} \\
\sigma_{0P} = \hat{R} \hat{\omega}_{JP}, \quad \sigma_{0Y} = \hat{R} \hat{\omega}_{3Y} \\
\]

where
\[ \sigma_{0P} = \text{Pitch Sliding Mode Variable}, \quad \sigma_{0Y} = \text{Yaw Sliding Mode Variable} \]
\[ \eta_{CP} = \text{Pitch Acceleration Command}, \quad \eta_{CY} = \text{Yaw Acceleration Command}, \quad K_G = \text{Guidance Gain} \]
\[ \hat{V}_C = \hat{V}_r = \text{Estimated Target-to-Interceptor Closing Velocity}, \]
\[ \hat{R} = \text{Estimated Target-to-Interceptor Range}, \quad \hat{\omega}_{JP} = \text{Estimated Pitch LOS Rate} \]
\[ \hat{X}_{BLOS} = \text{X component of the LOS unit vector in body frame (approximates cosine of the angle between the LOS and the body or interceptor velocity, assuming small AoA)} \]
\[ \eta_{1P} = \text{First Pitch Acceleration Command Term} \]
\[ \eta_{2P} = \text{Second Pitch Acceleration Command Term (Integral Term)} \]
\[ \eta_{1Y} = \text{First Yaw Acceleration Command Term} \]
\[ \eta_{2Y} = \text{Second Yaw Acceleration Command Term (Integral Term)}, \]
\[ \Delta t = \text{Guidance Update Period} \]

V. Simulations and Performance Analysis

This section presents the performance of the combinations of sliding mode guidance, observers, classical guidance, and Kalman filters. Figure 6 illustrates representative altitude vs. down-range trajectories for both target and interceptor for this study. The dispersion in the intercept points is due to the target maneuver. The initial results presented in the section apply the target maneuvers approximately 3.5 seconds prior to intercept. Homing guidance starts approximately 4 seconds prior to intercept.

### A. PN Guidance

Classical PN performance using sliding mode observers is compared with PN using a Kalman filter set. Figure 7 illustrates typical guidance commands and responses, as well as the maximum/minimum envelope about the mean interceptor to target acceleration ratio for a PN and sliding mode system. Where the interceptor to target maneuver ratio is defined as
where:

\[ A_R = \frac{\sqrt{A_{IZB}^2 + A_{TYB}^2}}{\sqrt{A_{ITZB}^2 + A_{ITYB}^2}} \]  

\[ A_R \] = Interceptor to Target Maneuver Ratio

\[ A_{IZB} \] = Z-Body Interceptor Acceleration - Normal to the Body

\[ A_{ITZB} \] = Z-Body Target Acceleration –Normal to the Body

\[ A_{TYB} \] = Y-Body Interceptor Acceleration - Normal to the Body

\[ A_{ITYB} \] = Y-Body Target Acceleration – Normal to the Body

Although the acceleration commands derived by the guidance law look rather noisy, the high bandwidth observer is a requirement to quickly respond to the target maneuver and to achieve an acceptable miss distance. For the 0.4Hz target maneuver frequency, target maneuver amplitude of 12gees (117.8m/s²) and a seeker angular noise of 100µr, 1σ, the mean maneuver ratio is less than two.

Figure 7: Typical PN Acceleration Commands and Responses For SOSM Based State Estimation; Maneuver Ratio Min, Max, Mean from 100 Monte Carlo Runs-Target Maneuver 12gees at 0.4Hz.

Figure 8 illustrates typical guidance commands and responses, as well as the maximum/minimum envelope about the mean interceptor to target acceleration ratio for a PN and Kalman filter system. The acceleration commands for the Kalman filter system are smoother, however the responses themselves are comparable. The acceleration ratios for both the Kalman filter system and the Sliding Mode Observer system are very similar.

Figure 9 illustrates side-by-side miss distance performance of PN using sliding mode observers and Kalman filters as guidance input sources. This figure indicates that the sliding mode filters perform better for lower weave frequencies while the Kalman filters perform better at higher frequencies. At lower frequencies the impact of target acceleration frequency on observed states provides a higher signal-to-noise ratio, after two integrations, than the higher frequency accelerations. The sliding mode observers as they are tuned here provide a higher bandwidth solution than the Kalman filters. This allows the sliding mode observer system to perform better at lower frequencies. However, at higher frequencies the Kalman filter does a bit better job of separating noise from target acceleration. Although this may seem counter intuitive, the higher frequency accelerations have a \(1/\omega^2\) amplitude effect on target position oscillations which the lower bandpass Kalman Filter can process more cleanly.
Figure 8: Typical PN Acceleration Commands and Responses For Kalman Filter Based State Estimation; Maneuver Ratio Min, Max, Mean from 100 Monte Carlo Runs—Target Maneuver 12gees at 0.4Hz.

Figure 9: PN Guidance Performance Against Weaving Targets With SOSM Observers and Kalman Filters

B. Second Order Sliding Mode Guidance

The performance of SOSM guidance (40) against target step and weaving maneuvers

Figure 10: Sliding Mode Guidance Against Step Target Maneuvers
Figure 11: Sliding Mode Achieved Normal Acceleration and Maneuver Ratio

maneuvers is shown here. Both sliding mode observers and Kalman filters were used to provide states for generating sliding mode guidance commands. Figure 10 illustrates that sliding mode guidance utilizing sliding mode observers performs slightly better than sliding mode guidance utilizing Kalman Filters. All three measures of miss distance for the sliding mode observer set stay within a meter, while only the CEP or 50 percentile miss remains within a meter at 9 gees for the Kalman filter set. The sliding mode observer set maintains a slightly smaller miss distance out to about 8gees.

Figure 11 illustrates the body normal acceleration components and the maneuver acceleration ratio. In this instance of a 12gee 0.4Hz target the maneuver ratio stays below 1.5 until very near endgame where it jumps to a value > 2.

Figure 12 compares sliding mode guidance miss distance for sliding mode observers and the Kalman filter set. The system with the Kalman filter set performs slightly better at higher frequency target maneuvers, while the Sliding Mode observer set performs better at lower frequencies, for the same reasons as the PN case. Time histories of an 8gee target step maneuver and interceptor acceleration are shown Figure 13. Since the plane of the target is random, there is acceleration on both y and z components of target acceleration. The target maneuver $\tau$ is 0.4 seconds.

Figure 14 illustrates the maximum, minimum envelope about the mean for the maneuver ratio from a 100 run Monte Carlo set. In this set the target maneuvers to 8gees and the interceptor applies sliding mode guidance and sliding mode observers. The mean maneuver ratio peaks around 2.5 just prior to the endgame transient.

Figure 12: Sliding Mode Performance Against Weaving Targets

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C. APN Guidance

Several sets of 6-DOF simulation Monte Carlo run sets were run with the APN guidance algorithm and standard 3 decoupled Kalman Filter set. No runs were made with sliding mode observers, as no sliding mode observers including target acceleration as an output state were included here. APN guidance performs well against step target maneuvers, however, its performance against weaving targets is inferior to both PN and Sliding Mode Guidance for scenario set used for this study. The performance against both step targets and weaving targets is shown in Figure 15. Ignoring the initial and endgame transient responses, the maneuver ratio against 8 gee step maneuver targets remained below 2.5, as shown in Figure 16.

VI. Conclusion

Smooth Second Order Sliding Mode Guidance and Second Order State Observation/Differentiation algorithms were presented and then implemented in a high fidelity 6-DOF simulation of a representative TBM interceptor. The scenario presented is in the low altitude TBM defense region.

PN guidance performance against spiral-maneuvering target is estimated using SOSC-based differentiator/observer versus Kalman Filter. The simulations indicate that the SOSC filters/observers perform better for lower weave frequencies while the Kalman filters perform better at higher frequencies. The SOSC observers provide a higher bandwidth solution than the Kalman filters. This allows the PN-based guidance with SOSC observer to perform better at lower frequencies. However, at higher frequencies the Kalman filter does a bit better job of separating noise from target acceleration.
SOSM guidance law is also tested using SOSM-based differentiator/observer versus Kalman Filter and demonstrates excellent performance against both step and spiral target maneuvers. Kalman Filter delivers better miss distance to SOSM-based guidance at high frequency maneuvering targets, while SOSM observers perform better at low frequencies. For weaving targets at maneuver levels from 8 to 12 gees, the combination of SOSM guidance with the decoupled Kalman Filter set had the best overall miss distance performance. However, the performance of SOSM guidance with SOSM Observers/Differentiators is comparable, while the SOSM Observers/Differentiators are simpler than Kalman Filter. SOSM guidance had the lowest and flattest miss distance against lateral step-maneuvering targets out to 8gees. Performance of SOSM-based guidance is better against step-maneuvering target using SOSM observers with an acceleration ratio less than 1.5.

APN guidance is tested for the sake of comparison with the SOSM and PN guidance. APN guidance performs well against step target maneuvers, however, its performance against weaving targets is inferior to both PN and SOSM guidance for scenario set used for this study. Apparently, APN guidance appeared to be more sensitive to the frequency content of the target maneuver as it is optimized for a step maneuver.

Finally, we could conclude that newly developed smooth SOSM guidance law and SOSM-based robust to noise observers/differentiators being tested on high fidelity 6-DOF Phoenix simulation package demonstrate a significant promise for kinetic energy missile-interceptors, providing for comparable and even superior performance as compared to PN and APN guidance operating with Kalman Filters.

For the future work, the SOSM-based autopilot* will be implemented and tested in 6 DOF missile model. The next effort will be dedicated to the high fidelity implementation and testing of the integrated SOSM guidance and autopilot*.
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