Design of thick-walled cylindrical vessels under internal pressure based on elasto-plastic approach

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ABSTRACT

By considering the Bauschinger effect and the yield criterion of Tresca, an exact elasto-plastic analytical solution for a thick-walled cylindrical vessel made of elastic linear-hardening material is derived. Having the working pressure and geometric dimensions of the vessel, the distribution of the hoop and equivalent stresses are optimized in the way that the distribution of stresses becomes smooth in the vessel wall. Based on two optimizing methods of the hoop and equivalent stresses, the best autofrettage pressure is determined. It is shown that this pressure is more than the working pressure and depends on the three following variables: Bauschinger effect, working pressure and geometric dimensions. In the next stage, the main task is to determine the wall thickness having the working pressure. To do this, two different design criteria namely; (i) optimizing the hoop stress distribution and (ii) assuming a suitable percent of yielding in the wall thickness are used. In the last step, for different types of structural materials under different working pressures, a number of different plots are given for the ratio of outer to inner radii and the best autofrettage pressure. It is shown that the design of vessels based on the elasto-plastic methods is much more economic than elastic methods. Also, it is seen that for a non-hardening material, the design of vessel is only done for the working pressure less than unit value.

1. Introduction

The ever increasing demands for axisymmetrical pressure vessels have high applications in chemical and nuclear industries, fluid transmitting plant, power plants and on the other hand in the military equipments. This wide spread need for the pressure vessels has turned the attention of many designers to this particular area of engineering researches. The progressive, world wide scarcity of some materials and also the high cost of their production have caused researchers not to confine their design to the customary elastic regime and have attracted their attention to the elastic–plastic approach which offers a more efficient use of such materials. In brief, increasing the strength to the weight ratio and extending the fatigue life are some of the main objectives of optimal design for the thick-walled cylinder. The optimization can be carried out primarily by generating a residual stress field in the cylinder wall, known as the autofrettage process. The study of this technique has been the subject of numerous researches for many years such as those given by Chen,[1,2] Franklin and Morrison[3], Zhu and Yang[4], Majzoobi et al.[5] and Bihamta et al.[6], moreover, Kargarnovin et al.[7–9] have discussed this technique further for non-hardening and hardening materials. For the linear-strain hardening material, however, little progress has been made that can represent satisfactory analytical solution for engineering use. Most of the earlier solutions for residual stresses were based on the assumption of elastic unloading and some investigators have considered a criterion known as the reverse yielding point.[10–13]

In this paper, at first, considering Bauschinger effect factor and different slopes for forward and reversed loading, a closed form solution for residual stresses in autofrettaged cylinders is obtained. In the next step a simple, accurate and reasonable analytical equation for determining the best autofrettage pressure with known cylinder dimensions and the working pressure is derived. Moreover, and conversely for cases where the working pressure and inner radius are known, the minimum acceptable thickness is determined. To make the outcome of this study more general, all obtained results and related plots are presented in the non-dimensional forms using Maple software version 8.

2. Formulations for the elastic–plastic loading

Consider a thick-walled cylinder with an inner radius “a”, and an external radius “b”, which is subjected to an inner pressure \( P_i \) (see Fig. 1). In order to make the solution method viewed in a more general way, the following dimensionless parameters are introduced:
During loading, based on the Tresca yield criterion, the stresses in the plastic zone \((1 < \rho < \rho_c)\) are given as follows [14]:

\[
\begin{align*}
S_t &= \frac{E_t}{1-m} \left( \left( \frac{c^2}{2} - P \right) \frac{\beta^2 - \rho^2}{\beta^2 - \rho_c^2} + 1 \right) \left( \frac{\beta^2 - \rho^2}{\beta^2 - \rho_c^2} \right), \\
S_0 &= \frac{E_0}{1-m} \left( \left( \frac{c^2}{2} - P \right) \frac{\beta^2 - \rho^2}{\beta^2 - \rho_c^2} + 1 \right) \left( \frac{\beta^2 - \rho^2}{\beta^2 - \rho_c^2} \right) + \frac{m}{1-m} \left( \ln \rho - P \right), \\
P &= \frac{\beta^2 - \rho^2}{\beta^2 - \rho_c^2} \left( \frac{1-m-v^2}{2} \right) + \frac{m}{1-m} \left( \ln \rho_c \right) \left( \frac{\beta^2 - \rho^2}{\beta^2 - \rho_c^2} \right).
\end{align*}
\]

where \(m = E_t/E_0\) (refer to Fig. 2) and \(P\) is given as follows:

\[
P = \frac{\beta^2 - \rho^2}{\beta^2 - \rho_c^2} \left( \frac{1-m-v^2}{2} \right) + \frac{m}{1-m} \left( \ln \rho_c \right) \left( \frac{\beta^2 - \rho^2}{\beta^2 - \rho_c^2} \right).
\]

Moreover, the stresses in the elastic zone \((\rho_c < \rho < \beta)\) are defined by following equations:

\[
S_t = \frac{P}{\beta^2} \left( 1 - \frac{\beta^2}{\rho^2} \right), \quad S_0 = \frac{P}{\beta^2} \left( 1 + \frac{\beta^2}{\rho^2} \right).
\]

### 3. Formulation for elastic–plastic unloading

When the pressure is removed from the cylinder that is under consideration, some residual stresses might be created. In order to calculate the residual stresses, it is necessary to superpose the obtained stresses in loading phase due to the internal pressure \(P\), and stresses caused during unloading phase \((-P)\).

It is clear that the equilibrium and compatibility relations during unloading are similar to those during loading phase. It is enough to substitute the constitutive relations into the compatibility equation and then combine the obtained result with the equilibrium equation. After integrating from the preceding relations, the following equations will come out [14]:

\[
\begin{align*}
S_t &= \frac{P}{\beta^2} \left( 1 - \frac{\beta^2}{\rho^2} \right), \quad S_0 = \frac{P}{\beta^2} \left( 1 + \frac{\beta^2}{\rho^2} \right), \\
C_1 &= \frac{1}{\beta^2 - 1} \left( P - \frac{\beta^2 - \rho^2}{\beta^2 - \rho_c^2} \right) \left( \frac{\beta^2 - \rho^2}{\beta^2 - \rho_c^2} \right) \left( \ln \rho_c \right) \left( \frac{\beta^2 - \rho^2}{\beta^2 - \rho_c^2} \right), \\
S' &= S_0 - S_t = \frac{2P}{\beta^2} \left( 1 - \frac{\beta^2}{\rho^2} \right) + \frac{P}{\beta^2} \left( 1 + \frac{\beta^2}{\rho^2} \right) + \frac{P}{\beta^2} \left( 1 + \frac{\beta^2}{\rho^2} \right),
\end{align*}
\]

where the superscript prime (′) denotes to the value symbol in the unloading phase.

Two different material hardening models, isotropic and kinematic hardening, are usually considered for reverse yielding. The isotropic hardening model is the simpler one to use, but it cannot incorporate the Bauschinger effect which is observed experimentally. To include the Bauschinger effect, Prager [15] suggested the kinematic hardening model. This model assumes that the yield surface translates as a rigid body in the stress space during the plastic deformation.

Consider a specimen of a specific material loaded in tension or compression into the plastic region. When the load is removed, the specimen is reloaded in the reverse direction elastically up to a point where yielding initiates. It has been observed that the yield stress obtained in the reloading or reversed direction is substantially less than the yield stress in the original direction. This phenomenon is called the Bauschinger effect. Therefore, in the case of kinematic hardening the reverse yielding is controlled by (see Fig. 3)

\[
S_0 = \begin{cases} 
-(1 + f)S & \rho \leq \rho_c, \\
-(S + 1) & \rho \geq \rho_c,
\end{cases}
\]

where \(S_0\) is the dimensionless equivalent stress in the reverse yielding and \(f\) is defined as the ratio of yield stress in reversed direction to the yield stress in tension. For example, Fig. 3 shows the Bauschinger effect factor (BEF) as a function of the percent of tensile over-stress \((\phi')\) for a type of material [14]. The graph shows a decrease of the BEF with increasing amount of tensile pre-strain up to approximately two percent then after this point onwards it becomes effectively constant. For fitting a function to the BEF curves similar to this graph, we assume the following form:

\[
f = \phi' + \phi' \exp(\gamma' \phi'),
\]

where the parameters \(\phi', \phi'\) and \(\gamma'\) are unknown constants that can be determined in the process of curve fitting. For a cylinder with
plastic radius $\rho_c$ during loading, this function is modified by the following mathematical form:

$$ f = \alpha + \delta \exp\left(\gamma (1/\rho_c^2 - 1/\rho^2)\right). $$

(8)

3.1. Reverse yielding

The maximum value for the applied pressure $P$, wherein the cylinder will have no reversed plastic flow during unloading, is defined with $P^*$. In order to determine $P^*$, it is enough to substitute $\rho_c = 1$ and $\varepsilon^p_r = 0$ into the two last of Eq. (5). Therefore,

$$ P^* = \frac{b^2 - 1}{b^2} (1 + f_1) \left[ m(1 - \nu^2) + \frac{1 - m}{1 - m \mu^2} \right]. $$

(9)

In general, if $P < P^*$ then unloading is entirely elastic and for $P > P^*$ yielding will occur during unloading. In order to analyze the effect of these changes, at first the values of two aforementioned pressures, i.e. $P$ and $P^*$ have to be equated. From equality expressions of $P$ and $P^*$ (Eqs. (9) and (13)), it is concluded that;

$$ m = - \frac{2 m_0 \rho_b (1 - \beta') \rho + f + 1}{f \rho_b^2 (v^2 - 1) + \rho_b^2 (1 + f - v^2) + \beta' (f - 2 m \rho_c)) - f - 1}. $$

(10)

Eq. (10) shows that the type of unloading depends on the variables $\rho_c$, $\beta$, $m$ and $f$. For example, curves in Fig. 3 indicate the interfaces between elastic unloading and elastic-plastic unloading for different values $m$ when $\beta = \beta'$, $\nu = 0.3$. It can be seen from Fig. 3 that for $[f \geq 0.35, m \geq 0.3]$, $[f \geq 0.4, m \geq 0.25]$, $[f \geq 0.5, m \geq 0.2]$, $[f \geq 0.6, m \geq 0.15]$ and $[f \geq 0.78, m \geq 0.1]$, the deformation mode during unloading is entirely elastic.

3.2. Elastic–plastic unloading

Now, suppose that the loading has been such that the internal pressure is larger than $P^*$ given by Eq. (10). During unloading, yielding will occur in the region $1 < \rho < \rho_c$ with $\rho_c < \rho_c$. Based on Tresca yielding criterion and linear strain hardening assumption, it follows from Fig. 2 that:

$$ \varepsilon^p_r = \varepsilon_{eq} = - \frac{1 - m'}{m'} (S' + S_0), $$

(11)

where $m'$ is the slope of strain-stress unloading curve. Substituting Eq. (11) into the last of Eq. (5), the function $S'$ can be obtained as:

$$ S' = \frac{2 m'(1 - \nu^2)}{1 - m' \nu^2} \left( C_1 - P \right) \left( \frac{1}{1 - \rho^2} \right) \left( 1 - \frac{1}{\rho^2} \right) + 1 - \frac{m'}{1 - m' \nu^2} \ln(\rho) \right) $$

(12)

Using the equilibrium equation of unloading phase for $1 < \rho < \rho_c$, it is concluded that;

$$ S_r = \int_1^\rho \frac{S}{\rho} d\rho + P. $$

(13)

Substituting Eq. (12) into Eq. (13) and doing some mathematical operations, the radial stress in the reverse yielding zone ($1 < \rho < \rho_c$) is obtained as:

$$ S_r = P + \frac{m'(1 - \nu^2)}{1 - m' \nu^2} \left[ C_1 - P \right] \left( 1 - \frac{1}{\rho^2} \right) + 1 - \frac{m'}{1 - m' \nu^2} \ln(\rho) \right) $$

(14)

Substituting Eq. (5) into Eq. (14), it is concluded that:

$$ S_r = P + \frac{m'(1 - \nu^2)}{1 - m' \nu^2} \left[ C_1 - P \right] \left( 1 - \frac{1}{\rho^2} \right) + 1 - \frac{m'}{1 - m' \nu^2} \ln(\rho) \right) $$

(15)

For linear kinematic hardening model (the pure kinematic), BEF can be specified as:

$$ f = \frac{2}{5} - 1. $$

(16)

Substituting Eq. (15) into Eq. (13), the radial stress is determined for the case of Prager kinematic hardening rule as follows:
and the hoop stress in the reverse yielding zone \((1 < \rho < \rho_c)\) is given by:
\[ S_q = S + S_v. \]  
(18)

For the zone \((\rho_c < \rho < \beta)\) the unloading phase is elastic. The stress distribution in this zone is similar to the elastic stress distribution of a cylinder with inner radius \(\rho_c\) and outer radius \(\beta\), subjected to an internal pressure of \(S_q\). Therefore, the stress distributions in the zone \((\rho_c < \rho < \beta)\) are determined as follows:
\[ S_q = -\frac{\rho_c^2 S_{P,c}}{\beta^2 - \rho_c^2} \left(1 - \frac{\beta^2}{\rho_c^2}\right), \]
\[ S_v = -\frac{\rho_c^2 S_{P,c}}{\beta^2 - \rho_c^2} \left(1 + \frac{\beta^2}{\rho_c^2}\right). \]  
(19)

In fact, \(S_{P,c}\) is equal to \(P\) of a cylinder with inner radius \(c\) and outer radius \(\beta\). Therefore, \(S_{P,c}\) can be determined by replacing “\(c\)” with “\(r\)” in Eq. \((17)\) as follows:
\[ S_{P,c} = \frac{\rho_c^2 - 1}{\beta^2 - 1} \left[m(1 - \nu^2) \frac{\beta^2}{1 - \mu^2} \frac{1 - m}{1 - m^2} \right]. \]  
(20)

Finally, the residual stresses are found by superposing the stress components obtained during loading and unloading phases:
\[ S_q^r = S_q + S_v, \]
\[ S_v^r = S_v + S_q. \]  
(21)

The superscript “\(r\)” indicates the residual stress component.

4. Optimizing the distribution of hoop and equivalent stresses

Now, suppose that the geometric dimensions of the cylinder and also the working pressure are known. The autofrettage pressure is an internal pressure which is applied to the cylinder before it is put into the work. Here, this pressure is determined based on two optimizing methods of the distribution of equivalent and hoop stresses.

4.1. Optimizing the distribution of hoop stress

Since, the hoop stress has a notable portion of the radial stress in the plastic zone, optimizing the hoop stress can be a suitable method to determining the autofrettage pressure. It is clear that the total hoop stress is the summation of residual hoop stress due to the autofrettage pressure and working hoop stress. Fig. 3 illustrates the variation of the non-dimensional hoop stress distribution in different initial elasto-plastic radius \(\rho_c\) for \(m = \mu = 0.1, \beta = 3, \rho_o = 0.75\) and \(f = 0.051 + 0.96 \exp(-0.9/\rho^2)\). In this figure, the points with negative maxima represent the position of \(\rho_c\) during unloading and points with positive maxima represent the position of \(\rho_c\) during loading. By precise examination of these curves, it can be noted that an envelope curve can be passed through all points having maximum value in each curve. Furthermore, it can be seen that this envelope curve has a minimum at a certain point. The pressure corresponding to this point is selected as the best autofrettage pressure. It can be calculated from Fig. 3 that the dimensionless autofrettage pressure is 0.95 in this case.
Fig. 6. Envelope curves passing through points of maximum hoop stress in different elasto-plastic boundaries for $m = m' = 0, \beta = 3, P_w = 0.75$.

Fig. 7. Optimum hoop stress curves vs. $\rho$ for $m = m' = 0, \beta = 3, P_w = 0.75$.

Table 1
<p>| Obtained result based on optimizing the hoop stress in a cylinder for $m = m' = 0, \beta = 3, P_w = 0.75$. |
|-------------------------------------------|-----------|-----------|</p>
<table>
<thead>
<tr>
<th>$\rho_{\text{opt}}$</th>
<th>$P_{\text{max}}$</th>
<th>$\sigma_{\text{max}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f = 1$</td>
<td>1.73</td>
<td>0.86</td>
</tr>
<tr>
<td>$f = 0.7$</td>
<td>1.77</td>
<td>0.88</td>
</tr>
<tr>
<td>$f = 0.4$</td>
<td>1.85</td>
<td>0.91</td>
</tr>
<tr>
<td>$f = 0.2$</td>
<td>1.97</td>
<td>0.94</td>
</tr>
</tbody>
</table>

Fig. 8. Envelope curves passing through points of maximum hoop stress in different elasto-plastic boundaries for $m = m' = 0.1, \beta = 3, P_w = 0.75$.

Fig. 9. Optimum hoop stress curves vs. $\rho$ for $m = m' = 0.1, \beta = 3, P_w = 0.75$.

Table 2
<p>| Obtained result based on optimizing the hoop stress in a cylinder for $m = m' = 0.1, \beta = 3, P_w = 0.75$. |
|-------------------------------------------|-----------|-----------|</p>
<table>
<thead>
<tr>
<th>$\rho_{\text{opt}}$</th>
<th>$P_{\text{max}}$</th>
<th>$\sigma_{\text{max}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kinematic</td>
<td>1.82</td>
<td>0.95</td>
</tr>
</tbody>
</table>
stresses due to the best autofrettaged pressure in different cases for a non-hardening \((m = m' = 0)\) and hardening material \((m = m' = 0.1)\), respectively.

### 4.2. Optimizing the distribution of equivalent stress

In the same way as the previous section, the best autofrettage pressure is determined based on optimization of equivalent stress in the vessel’s wall. Fig. 10 shows the equivalent stress distributions for different initial elasto-plastic radius, \(q_c\), for a cylinder with \(m = m' = 0.1\), \(\beta = 3\), \(P_w = 0.75\) and \(f = 0.051 + 0.96\exp(-0.9/\rho^2)\). Again, it can be observed that an envelope curve can be passed through all points having maximum value in each curve. It can be seen that this curve has a minimum at a certain point. Similarly, the best autofrettage pressure is selected for \(q_c\) which the maximum \(S_e\) has the lowest value rather than the other elasto-plastic radii. It can be calculated from Fig. 10, the dimensionless autofrettage pressure is 1.19 in this case.

Therefore, the position of minimum point of the envelope curve \(\rho_{c,opt}\) is found by applying the minimization rule on equivalent stress as:

\[
S_{e,\text{total}} = S_{e,q_c} + \frac{2P_w}{\beta^2 - 1} \left( \frac{\beta^2}{\rho_c^2} \right). \tag{25}
\]

Using Eq. (25), for the case of entire elastic unloading the position of \(\rho_{c,opt}\) becomes

\[
\rho_{c,opt} = \frac{e^{-m^2/2P_w}}{1 - \beta^2}. \tag{26}
\]

Fig. 10. The equivalent stress distribution vs. \(\rho\) in the different initial elasto-plastic boundaries for \(m = m' = 0.1\), \(\beta = 3\), \(P_w = 0.75\) and \(f = 0.051 + 0.96\exp(-0.9/\rho^2)\).

Figs. 11 and 12 show the effect Bauschinger factor \((f = 0.3, 0.6\) and 1) on the generated envelope curve passing through all maxima of equivalent stress for a non-hardening and hardening material, respectively. Figs. 13 and 14 illustrate the distribution of equivalent stress due to the best autofrettaged pressure for different values of BEF for a non-hardening and hardening material, respectively. Tables 3 and 4 display a comparative study between the obtained results based on optimizing the equivalent stress in the different cases.

### 5. Thickness optimization

Hitherto, the geometrical dimensions of the cylinder and working pressure were known and we could determine \(P_{opt}\). Now, it is assumed that \(P_w\) is known and we want to determine the optimum thickness accompanied with the best autofrettage pressure. Here, the amount of consumed material is minimized based on two design criteria, i.e. (i) optimizing the hoop stress, (ii) assuming a suitable percent for the yield zone in the wall thickness. Based on (i)-criterion, we have

\[
\frac{\partial}{\partial \rho} (S_{h,max}) = 0. \tag{27}
\]
The process initiates from the inner radius. Therefore, according to step is loaded by a predefined working pressure, another yielding the first step undergoes an autofrettage process and then in the next

\[ S_{v1} + 2p_w\beta^2/(\beta^2 - 1) = S_{v1} \]  

(28)
in which, \( S_{v1} \) is the value of the new yielding stress at inner radius. If the unloading is elastic, then \( S_{v1} = S_{e1} \), otherwise \( S_{v1} = -((1 + f)(S_{e1} + S_{e2})) \).

In the autofrettage process and when the working pressure is applied, yielding initiates at the points on the inner surface of the cylinder. In order to avoid such kind of incident one can introduce an additional constraint. To do this, let us determine the values of \( \rho_{c,\text{opt}} \) and \( \beta \) by solving following equations:

\[
\begin{align*}
\frac{\partial}{\partial \rho}(S_{\text{max}}) &= 0 \quad \text{condition 1} \\
S_{e1} + 2p_w\beta^2/(\beta^2 - 1) &= S_{v1} \quad \text{condition 2} \\
S_{v1} &\leq 1.7 \quad \text{condition 3}
\end{align*}
\]

(29)

where the constraint of \( S_{v1} \leq 1.7 \) is used to avoid the failure in the inner layer.

In the case of elastic-perfectly plastic behavior, condition 2 of Eq. (29) gives:

\[ p_w\beta^2/(\beta^2 - 1) \leq 1. \]  

(30)

It is notable that the inequality in Eq. (30) has a result only for \( p_w \leq 1 \), therefore, for non-hardening materials, the system of Eq. (29) can be solved when \( p_w \leq 1 \).

If the unloading is entirely elastic, then the system of Eq. (29) results in:

\[
\begin{align*}
\beta &= \frac{\sqrt{2} - p_w + p_wm^2 - \rho_{c,\text{opt}}q_c(\rho_{c,\text{opt}}^2 + m^2p_w^2 - 1 - m^2p_w^2^{1/2})} {2\sqrt{1 - m^2p_w^2}} + \rho_{c,\text{opt}}^2 + m^2p_w^2^{1/2} \rho_{c,\text{opt}}^2 - \rho_{c,\text{opt}}^2} \\
p_w &= \frac{\sqrt{2} - 1 - \rho_{c,\text{opt}}q_c(\rho_{c,\text{opt}}^2 + m^2p_w^2 - 1 - m^2p_w^2^{1/2})} {2\sqrt{1 - m^2p_w^2}} + \frac{1 - m^2p_w^2^{1/2}} {2\sqrt{1 - m^2p_w^2}} \rho_{c,\text{opt}}^2 \rho_{c,\text{opt}}^2 - \rho_{c,\text{opt}}^2}
\end{align*}
\]

(31)

If the unloading is not elastic, then the system of Eq. (29) ought to be solved numerically.

Fig. 13 shows the variations of the dimensionless radius \( \beta \) and the dimensionless optimum autofrettage pressure, \( P_{opt} \), versus working pressure \( p_w \). The jump points in these curves show that reverse yielding occur in the unloading phase due to the optimum autofrettage pressure. Using Eq. (61), it can be shown that the new yielding stress at inner radius, \( S_{v1} \), decreases by inclusion of BEF. This in turn will cause an increase in the value of the wall thickness.

Based on (ii)-criterion, in order to determine \( \beta \) and \( P_{opt} \), it is necessary to choose an appropriate value for the percentage of yielded zone. If the percent value for this yielded zone is taken to be 100% then the common boundary between elastic and plastic regions will shift towards the external radius. This means that the whole wall thickness undergoes plastic deformation and this would not be a right design principle. On the other hand, if this percent value for this yielded zone is taken to be 0%, the common boundary will shift towards the internal radius; again this would not be suitable design criterion, because in this case the wall thickness becomes very large. In this study, we consider the depth of this yielded zone to be 75% from the inner surface. Therefore,

\[ \rho_{c,\text{opt}} = 1 + 0.75(\beta - 1) \]  

(32)

Likewise, Eq. (32) and two last conditions in Eq. (29) compose a set of equations for determining \( \beta \) and \( P_{opt} \) based on (ii)-criterion.

Fig. 13 displays the variations of the dimensionless values \( \beta \) and \( P_{opt} \) obtained based on (ii)-criterion versus working pressure for

### Table 3

Obtained result based on optimizing the equivalent stress in a cylinder for \( m = m' = 0.1, \beta = 3, P_w = 0.75 \).

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>( P_{opt} )</th>
<th>( P_{unl} )</th>
<th>( S_{\text{eq}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Isotropic</td>
<td>2.13</td>
<td>0.99</td>
<td>0.83</td>
</tr>
<tr>
<td>( f = 0.6 )</td>
<td>2.21</td>
<td>1.00</td>
<td>0.83</td>
</tr>
<tr>
<td>( f = 0.3 )</td>
<td>2.27</td>
<td>1.01</td>
<td>0.71</td>
</tr>
</tbody>
</table>

### Table 4

Obtained result based on optimizing the equivalent stress in a cylinder for \( m = m' = 0.1, \beta = 3, P_w = 0.75 \).

<table>
<thead>
<tr>
<th>( \rho_{c,\text{opt}} )</th>
<th>( P_{unl} )</th>
<th>( S_{\text{eq},\text{opt}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f = 1 )</td>
<td>2.455</td>
<td>1.204</td>
</tr>
<tr>
<td>( f = 0.6 )</td>
<td>2.43</td>
<td>1.2</td>
</tr>
<tr>
<td>( f = 0.3 )</td>
<td>2.39</td>
<td>1.18</td>
</tr>
</tbody>
</table>

Kinematic: 1.91 0.993 0.961

<table>
<thead>
<tr>
<th>F</th>
<th>0.3 2.39 1.18 0.77</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f = 0.6 )</td>
<td>2.43</td>
</tr>
<tr>
<td>( f = 0.3 )</td>
<td>2.39</td>
</tr>
</tbody>
</table>

PDF 1.7 is used to avoid the failure in the inner layer.
In this paper, by considering BEF the optimal design of thick-walled cylindrical vessels was studied based on an elastic–plastic approach. For evaluation purposes, the material behavior was assumed to be a linear strain hardening that obeys Tresca’s yield condition with associated flow rule. The obtained results were presented in a dimensionless form, and therefore, can be used for any loading conditions and thickness ratios. In all studied cases, the results show that the residual hoop and equivalent stresses are highly compressive at the inner surface and tensile at the outer surface. By prescribing the working pressure and dimensionless radius $\beta$, the distribution of hoop and equivalent stresses across thickness were optimized. Based on two optimizing methods of the distribution of equivalent and hoop stresses, the best autofrettage pressure was determined. It was observed that presented methods for determining of $P_{\text{opt}}$ are very simple, applicable and reasonable compared to other existing methods. Also, it was seen that the autofrettage pressure is greater than the working pressure and depends on BEF, the value of working pressure and the geometrical dimensions of the cylinder. Moreover, if the unloading is entirely elastic then the explicit expressions for determination of $P_{\text{opt}}$, $F_{\text{opt}}$ are derived.

In the following, for a given value of the working pressure, the minimum value of the geometrical parameter, $\beta$, and autofrettage pressure, $P_{\text{opt}}$, were obtained. For doing this, by introducing two design criteria that are, (i) optimizing the hoop stress, (ii) assuming a suitable percent for the yield zone in the wall thickness, the minimum thickness was calculated for these vessels. It was shown that the calculated wall thickness based on the elasto-plastic methods is less than that of obtained using the criterion of the initiation of yielding in the inner radius. In the case of elastic–perfectly plastic ($m=0$), when $p_{\text{w}}=1$ the geometric dimensions of the vessel approach to the infinity. It was seen that by decreasing the Bauschinger effect factor, the optimum autofrettage pressure and the geometric parameter will increase. Finally, when unloading is elastic, then $P_{\text{opt}}=p_{\text{w}}$ therefore, the slope of $P_{\text{opt}}-p_{\text{w}}$ curve is unit and $P_{\text{opt}}$ has nonlinear behavior when the unloading is elastic–plastic.

6. Conclusions

In this paper, by considering BEF the optimal design of thick-walled cylindrical vessels was studied based on an elastic–plastic approach. For evaluation purposes, the material behavior was assumed to be a linear strain hardening that obeys Tresca’s yield condition with associated flow rule. The obtained results were

References


