Redundancy resolution and control of a novel spatial parallel mechanism with kinematic redundancy

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1. Introduction

Parallel manipulators are extensively used in applications where good dynamic performances are needed [1,2]. In parallel manipulators, the actuators are mounted in the base leading to the reduction of the inertia of the moving parts. Furthermore, they have high load carrying capacity because the external loads are shared by the actuators. In contrast with serial robots, the errors in the actuators of parallel robots are not cumulative and the positioning accuracy of the end-effector of parallel robots is only slightly affected by errors in the actuators.

Parallel manipulators can be found in many applications, such as airplane simulators, adjustable articulated trusses, mining machines, and walking machines. However, parallel manipulators suffer from the problem of a large number of singularities within their workspaces. The mentioned drawback can be overcome by kinematic or actuator redundancy [3,4]. A detailed review of redundant parallel mechanisms is given by Luces et al [5].

This research has focused on the kinematically redundant manipulators having the following improved performance over non-redundant ones [6,7]:

1. Due to redundancy, there are infinite solutions for the inverse kinematics of the redundant manipulators and consequently, they can be actuated in such a way to avoid singularities and to reduce the actuator forces.
2. The forces required for tracking a given trajectory are lower than those in the non-redundant manipulators [8,9].
3. In many industrial applications, such as pick and place, the time must be minimized. Kinematically redundant mechanisms can be actuated in such a way to minimize time while tracking a given trajectory.

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In literature, different types of kinematically redundant mechanisms were introduced and the inverse kinematics, workspace and singularity analyses were investigated. Three new types of kinematically redundant parallel mechanisms were proposed by Wang and Gosselin [10]. The velocity equations were derived and the singularity loci of the proposed redundant mechanisms were compared with those of similar non-redundant ones. Furthermore, a family of kinematically redundant planar parallel manipulators was presented based on 3–PRRR architectures by Ebrahimi et al [11]. The inverse kinematics of the proposed mechanisms were studied and their reachable and dexterous workspaces were obtained. Gosselin et al [12] have introduced a family of singularity free kinematically redundant planar parallel mechanisms that have unlimited rotational capability. Gosselin and Schreiber [13] introduced a novel architecture of kinematically redundant manipulator based on the well-known Gough–Stewart platform. They used Grassmann geometry to illustrate how all singularities could be avoided. In [14], a planar redundant mechanism categorized as 3–PRPR type is introduced. The structure of the proposed mechanism is elaborated and its constant orientation and reachable workspace is studied. Landur and Gosselin [15] have introduced a new architecture of spherical parallel robot, which has a theoretical workspace much larger than that of the counterpart non-redundant mechanism. The singularity locus are addressed as a function of the architectural parameters from a geometric point of view. Schreiber and Gosselin [16] have proposed two general architectures of kinematically redundant planar parallel mechanisms. The kinematics of the introduced mechanisms are studied and an analytical method is proposed for workspace determination.

In spite of promising properties of kinematically redundant mechanisms, there are some disadvantages such as [6-7]:

1. In contrast to non-redundant mechanisms, kinematic analysis of redundant mechanisms is more challenging. For kinematically redundant parallel mechanisms, the inverse displacement has infinity of solutions for a given pose of the moving plate.
2. Redundant mechanisms have more actuators than similar non-redundant ones which make the actuation and control of these mechanisms difficult and expensive.

To address the aforementioned challenges, different redundancy resolution approaches for motion planning of kinematically redundant parallel mechanisms are proposed. Redundancy resolution is defined as the selection of a single kinematic configuration among a set of many possible ones [17]. Motion planning for kinematically redundant parallel mechanisms is defined as finding the actuation schemes of the actuators such that the end-effector of the mechanism to track a given trajectory [18]. In what follows, the proposed motion planning strategies in the literature are elaborated:

Chá et al. [19] introduced a method to obtain the required redundant active prismatic joint variables ranges for singularity-free trajectories.

Ebrahimi et al. [20,21] have presented a method to actuate kinematically redundant mechanisms called point-to-point motion planning (PPMP) in which, the objective function was defined based on the proximity to the singular configurations. In PPMP, the next displacement of each redundant actuators is obtained based on their current values. The method can be implemented online.

A new approach called overall motion planning (OMP) was introduced by Carretero et al. [18,22] in which the manipulator performance was optimized while the entire trajectory of the end-effector at once was considered.

Ruggiu and Carretero [23] proposed a new actuation strategy based on actuator’s accelerations. The method was applied to a 3–PRPR redundant planar parallel manipulator. In this method, the position of the redundant actuators are considered as the search variable, for any selected value of the search variables, the velocity and acceleration of redundant actuators are obtained using the time history of the actuator displacements. The velocity and acceleration of non-redundant actuators are obtained using kinematic equations. In each step, the displacement of redundant actuators are obtained to minimize the 2-norm of acceleration vector.

Nouri et al. [24] have proposed an optimal motion planning algorithm to solve the mentioned challenges for actuation of parallel mechanisms with kinematic redundancy. In this method, the velocity of redundant actuators are considered as the search variables and are obtained in such a way to minimize the 2-norm of velocity vector. The position of the redundant actuators are obtained by numerical methods. In [17], a redundancy resolution strategy via Differential Dynamic Programing is used for actuating a 3–PRRR planar redundant manipulator. The main implication emerging from this research is to propose an online motion planning algorithm considering the full dynamics of the mechanism. The proposed method is used for motion planning and control of the novel spatial kinematically redundant mechanism. The general configuration of the introduced mechanism is explained and its kinematics, Jacobian matrix and dynamic equations are derived and the movement of the mechanism in tracking a given trajectory is simulated.

2. Redundancy resolution

Non-redundant mechanisms have limited solutions for their actuations. Furthermore, in such mechanisms switching from one solution to another one is impossible. But, there are infinite solutions for the actuation scheme of the kinematically redundant mechanisms. Therefore, it is possible to select a solution for the better performance of such mechanisms. Redundancy resolution consists of finding extra actuators movement to improve the performance of the mechanism [25]. For the purpose of redundancy resolution, two issues can be addressed. The first one is to approximate a smooth path for the redundant actuators and the other one is the selection of a cost function to determine the optimal path. In the following, the path parameter representation and the cost function selection for an online redundancy resolution are discussed.
2.1. Path parameter representation

In this work, instead of determining the position of the redundant actuators at each time, an optimization problem is defined and solved in a limited number of specific times \((t_0, t_1, \ldots, t_N)\), then the smooth motion is generated by means of interpolating functions and incorporating kinematic constraints.

The displacement of each redundant actuator is approximated with \(N\) adjacent cubic curve segment as depicted in Fig. (1), whose \(N+1\) points are named control points. The length of time intervals is assumed equal and is depicted by \(\Delta t\). By altering the position of control points the path of each redundant actuators can be modified.

The movements of each redundant actuator in the \(i\)th time interval, \(t_{i-1} \leq t \leq t_i\), (\(i = 1, 2, \ldots, N\)) can be approximated by a uniform cubic B-spline \([26]\) as

\[
q(\tau) = U NB,
\]

in which, the vectors \(U\), \(B\) and the matrix \(N\) are

\[
U = \begin{bmatrix} \tau^3 & \tau^2 & \tau & 1 \end{bmatrix},
\]

\[
B = \begin{bmatrix} q_{i-2} & q_{i-1} & q_i & q_{i+1} \end{bmatrix}^T,
\]

\[
N = \frac{1}{2} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 2 & -5 & 4 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix},
\]

where

\[
\tau = (t - t_{i-1})/\Delta t, \quad t_{i-1} \leq t \leq t_i,
\]

where, \(q_{i-2}\), \(q_{i-1}\), \(q_i\) and \(q_{i+1}\) are the four adjacent control points.

The velocity and acceleration of each actuator can be written by the time derivative of Eq. (1) as

\[
\dot{q}(\tau) = U' NB,
\]

\[
\ddot{q}(\tau) = U'' NB,
\]

in which

\[
U' = \frac{1}{\Delta t}\begin{bmatrix} 3\tau^2 & 2\tau & 1 & 0 \end{bmatrix},
\]

\[
U'' = \frac{1}{\Delta t^2}\begin{bmatrix} 6\tau & 2 & 0 & 0 \end{bmatrix}.
\]

In each time interval, four adjacent control points should be obtained to determine the trajectory of each redundant actuator. The position, velocity and acceleration of the actuators are written in terms of control points which can be obtained to minimize the objective function.
2.2. Cost function and constraints

The control points in each time interval should be obtained to improve the performance indices. Different accuracy indices can be used for finding the optimal actuation of redundant mechanisms. In order to include the dynamics of the mechanism, the two norms of the actuator forces are considered as the objective function. For a singularity-free trajectory, the condition number of the Jacobian matrix should be considered. Parallel mechanisms have three types of kinematic singularities [27,28]. The most problematic kind of singularities occurs when the determinant of the direct Jacobian matrix, $J_x$, is zero. This kind of singularity is inside the workspace of the mechanism and the moving plate can locally move even though all the actuated joints are locked.

The condition number of the direct Jacobian matrix can be considered as the index in order to avoid singularity.

$$\eta = \text{cond}(J_x).$$

(10)

It is worth mentioning that, the condition number close to one is a desired characteristic. This index is zero in singular configurations.

The stroke limitation of each redundant actuator can be considered as a constraint. Also, the velocity and acceleration of each redundant actuator can be constrained by admissible choices for control points.

2.3. Proposed algorithm

It is assumed that the class of kinematically redundant mechanisms considered in this research is obtained by adding one redundant actuator to at least one of the chains of a non-redundant parallel mechanism having one actuator in each of its chains.

In sequence, an online novel redundancy resolution is proposed. The method is general and can be easily extended to actuate different classes of kinematically redundant mechanisms. The proposed method is summarized in the following steps.

1. First, the number of actuators ($n$) and the degrees of freedom of the end-effector ($m$) are specified and the $n$-m redundant actuators are selected.
2. The desired trajectory of the end-effector is specified and the total time for tracking the path, $T$, is divided into $N$ time steps with time interval $\Delta t = T/N$.
3. In this step, the joint displacement of each redundant actuator in each time interval is determined using the proposed redundancy resolution. In the $i$th time interval, the control points $q_{i-2}$, $q_{i-1}$ and $q_i$ are determined from the previous time interval. The control point $q_{i+1}$ is considered as the search variable and must be obtained to determine the trajectory of the redundant actuator. Considering the condition number of the Jacobian matrix at $i+1$th control point and the two norms of the actuator forces at $i$th control point, as the objective function in the $i$th time interval, the optimization problem is formulated as

$$\min \text{cond}(J_x(t_{i+1})) + \sum_{i=1}^{n} \lambda_i u_i^2(t_i),$$

Search Variable : $q_{i+1}$

subject to:

$$q_{\text{min}} < q < q_{\text{max}}, \quad \dot{q}_{\text{min}} < \dot{q} < \dot{q}_{\text{max}}$$

(12)

in which, $u_i$ is the force in the $i$th actuator and $\lambda_i$ are the weighting coefficients.

1. Given the desired motion of the moving plate and the redundant actuators, a controller can be designed for motion control of the mechanism.
2. Steps 3 and 4 are repeated till the last time interval of the trajectory.

To implement the optimization problem, the following remarks are in order

**Remark 1.** For the first time interval, the position of control points, $q_0$ and $q_1$, are obtained in such a way to be far from singular configurations and $q_{-1}$ is selected equal to $q_0$. Furthermore, for the last time interval, it is assumed that $q_N = q_{N+1}$, for each redundant actuator.

**Remark 2.** In the applications such as milling that the mechanism interacts with the environment, the external forces should be taken into account for redundancy resolution. In the proposed method, estimating the external forces, the joint displacement are obtained in such a way to minimize the actuator forces and avoid singular configurations.

**Remark 3.** The optimization problem for determining the desired trajectory in each time interval is solved in the previous time interval.

**Remark 4.** The duration of time intervals should be selected with care. In the applications with high accuracy, the control law should be refreshed quite high and the duration of the time intervals should decrease.
Remark 5. The only limitation on the proposed method is that the run time for solving the optimization problem should be less than the interval times for a real time redundancy resolution.

4. The proposed mechanism with PUPS chain

The Stewart–Gough platform has been studied by many researchers. It consists of a moving platform connected to the base via six branches. Each branch is composed of a passive universal joint, followed by an actuated prismatic joint and by a spherical joint which is attached to the moving platform. The proposed architecture of the redundant mechanism is based on Stewart–Gough Platform. The mechanism has three prismatic joints on the base and six lateral prismatic actuators which control the end effector with six degrees of freedom. The added redundancy might be exploited to increase achievable “tilt angle” at the platform, which could be of interest for specific applications such as milling and welding. The Cad model of the proposed kinematically redundant mechanism is shown in Fig. (2). The mechanism consists of three base guides on which two universal joints are mounted. Furthermore, the mechanism has six lateral prismatic actuators connected at one end to the moving plate by a spherical joint and at the other end to the prismatic actuators on the base by a universal joint.

5. Kinematic modeling

The base and the moving plate of the proposed mechanism are depicted in Fig. (3), and various geometric parameters are presented.

In order to analyze the proposed mechanism, as shown in Fig. (3)a reference frame B(x, y, z) is attached to the intersection of the axes of prismatic actuators on the base, while x-axis points to the direction of the first prismatic actuator and z-axis is perpendicular to the base. Furthermore, the moving frame, P(x’, y’, z’) is attached to the geometric center of the moving plate in such a way that x’ and y’-axes lay in the plane of the moving plate and x’-axis points to the midpoint of a1a2.

5.1. Kinematic equations of the moving plate

The position of an arbitrary point on the end effector can be expressed by the position vector of the origin of body fixed frame P relative to the reference frame B and the set of Euler angles (ψ, θ, φ) in a 3-2-1 sequence defining the orientation of the end effector in B. The transformation matrix from P to B is

$$R = R_z(\psi)R_y(\theta)R_x(\phi) = \begin{bmatrix}
c\psi c\theta & -s\psi c\phi + c\psi s\theta s\phi & s\psi s\phi + c\psi s\theta c\phi \\
s\psi c\theta & c\psi c\phi + s\psi s\theta s\phi & -c\psi s\phi + s\psi s\theta c\phi \\
-s\theta & c\theta s\phi & c\theta c\phi
\end{bmatrix}. \quad (13)$$

where, c and s stand for cosine and sine.

The position vector of node $a_i^j$ with respect to the reference frame B can be written as

$$a_i^j = p + \bar{R}u_i^j. \quad (14)$$
where, vector $\mathbf{p} = [p_x, p_y, p_z]^T$ is the position vector of the origin of frame A and $\bar{\mathbf{u}}_i^j$ is the position of point $a_i^j$ presented in P.

$$\bar{\mathbf{u}}_i^j = \mathbf{Z}_i^T \mathbf{a} \begin{pmatrix} -1 \\ b \\ 0 \end{pmatrix}.$$  

where, constants a and b are shown in Fig. (3). The transformation matrix $\mathbf{Z}_i$ is defined as

$$\mathbf{Z}_i = \begin{bmatrix} c\gamma_i & -s\gamma_i & 0 \\ s\gamma_i & c\gamma_i & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where

$$\gamma_1 = 0, \gamma_2 = 2\pi/3, \gamma_3 = 4\pi/3.$$  

The position vector $\bar{\mathbf{u}}_i^j$ can be presented in the frame B using the transformation matrix as

$$\mathbf{u}_i^j = \mathbf{R}_1 \bar{\mathbf{u}}_i^j.$$  

The angular velocity of the moving plate with respect to the reference coordinate frame can be written as [29]

$$\mathbf{\omega} = (-s\psi \dot{\phi} + c\theta c\psi \dot{\phi}) \mathbf{x} + (c\psi \dot{\theta} + c\theta s\psi \dot{\phi}) \mathbf{y} + (\dot{\psi} - s\theta \dot{\phi}) \mathbf{z}.$$  

In addition, the angular acceleration is found by time derivative of Eq. (19). The angular velocity and acceleration of the moving plate can be presented in a matrix form as

$$\dot{\mathbf{\omega}} = \mathbf{R}_1 \dot{\Theta}, \quad \ddot{\mathbf{\omega}} = \mathbf{R}_1 \ddot{\Theta} + \mathbf{R}_2 \Theta,$$

where, $\mathbf{\Theta} = [\psi, \theta, \phi]^T$ shows the orientation of the moving plate and matrices $\mathbf{R}_1$ and $\mathbf{R}_2$ are obtained as

$$\mathbf{R}_1 = \begin{bmatrix} 0 & -s\psi & c\theta c\psi \\ 0 & c\psi & c\theta s\psi \\ 1 & 0 & -s\theta \end{bmatrix},$$

$$\mathbf{R}_2 = \begin{bmatrix} 0 & -c\psi \dot{\psi} & -s\theta c\psi \dot{\phi} - c\theta s\psi \dot{\phi} \\ 0 & -s\psi \dot{\psi} & -s\theta s\psi \dot{\theta} + c\theta c\psi \dot{\phi} \\ 0 & 0 & -c\theta \dot{\phi} \end{bmatrix}.$$  

The velocity of node $a_i^j$ can be obtained by time derivative of Eq. (14)

$$\dot{\mathbf{a}}_i^j = \dot{\mathbf{p}} + \mathbf{\omega} \times \mathbf{u}_i^j.$$  

The above relation can be represented as

$$\dot{\mathbf{a}}_i^j = [I_{3x3} \quad -\bar{\mathbf{u}}_i^j \mathbf{R}_1] \begin{bmatrix} \dot{\mathbf{p}} \\ \Theta \end{bmatrix}.$$
where, \( \hat{\mathbf{u}}_i \) is the skew symmetric matrix associated with vector \( \mathbf{u}_i \).

The acceleration of node \( \mathbf{a}_i \) can be obtained by time derivative of Eq. (23)

\[
\mathbf{a}_i = \mathbf{p} + \omega \times (\omega \times \mathbf{u}_i) + \ddot{\omega} \times \mathbf{u}_i. \tag{25}
\]

Using Eq. (20), the following relation can be obtained for the acceleration of node \( \mathbf{a}_i \)

\[
\mathbf{a}_i = \ddot{\mathbf{p}} - \ddot{\mathbf{u}}_\mathbf{R}_1 \Theta + (\ddot{\omega})^2 \mathbf{u}_i - \ddot{\mathbf{u}}_\mathbf{R}_2 \Theta. \tag{26}
\]

in which, \( \ddot{\omega} \) is the skew symmetric matrix associated with \( \omega \).

Eq. (26) can be rewritten more compactly as

\[
\mathbf{a}_i = \left[ \mathbf{I}_{3 \times 3} - \ddot{\mathbf{u}}_\mathbf{R}_1 \right] \ddot{\mathbf{p}} + \mathbf{v}_i', \tag{27}
\]

where, \( \mathbf{v}_i' \) is defined as

\[
\mathbf{v}_i' = (\ddot{\omega})^2 \mathbf{u}_i - \ddot{\mathbf{u}}_\mathbf{R}_2 \Theta. \tag{28}
\]

5.2. Kinematic analysis of the mechanism with \textit{PUPIS} limbs

The position vector of each universal joint on the base actuators is

\[
\mathbf{b}_i = [l_i \quad -(-1)^i \mathbf{h} \quad 0]^T. \tag{29}
\]

where, the length of each prismatic actuator on the base are designed by \( l_i \).

Vector \( \mathbf{b}_i \) can be represented in the reference frame as

\[
\mathbf{b}_i = \mathbf{Z}_i (\gamma_i) \mathbf{b}_i'. \tag{30}
\]

The velocity and acceleration vectors, \( \dot{\mathbf{b}}'_i \) and \( \ddot{\mathbf{b}}'_i \) can be obtained as

\[
\dot{\mathbf{b}}'_i = \mathbf{h}'(\gamma_i) \dot{l}_i, \quad \ddot{\mathbf{b}}'_i = \mathbf{h}'(\gamma_i) \ddot{l}_i, \tag{31}
\]

where

\[
\mathbf{h}'(\gamma_i) = \begin{bmatrix} c\gamma_i & s\gamma_i & 0 \end{bmatrix}. \tag{32}
\]

Furthermore, the position vector of \( a_i \) can be written in terms of the limb parameters as

\[
\mathbf{a}_i = \mathbf{b}_i' + l_i' \mathbf{s}_i', \tag{33}
\]

where, \( l_i' \) is the length of lateral prismatic actuators and the unit vector along the \( i \)th leg axis is designated by \( \mathbf{s}_i' \).

In what follows, analytical expressions for angular velocities of the lateral prismatic actuators, \( \Omega_i' \) and the time derivative of lengths, \( \ddot{l}_i' \) are obtained. The velocity vector of point \( a_i \) can be derived as

\[
\dot{\mathbf{a}}_i = \dot{\mathbf{b}}'_i + l_i' \mathbf{s}_i' + l_i' \Omega_i' \times \mathbf{s}_i'. \tag{34}
\]

By cross multiplication Eq. (34), by \( \mathbf{s}_i' \) and assuming that the angular velocity of the \( i \)th lateral prismatic actuator is perpendicular to \( \mathbf{s}_i' \) [30], i.e. \( \Omega_i' \cdot \mathbf{s}_i' = 0 \) an analytical expression for \( \Omega_i' \) is obtained as

\[
\Omega_i' = \frac{1}{l_i'} \mathbf{s}_i' \times (\dot{\mathbf{a}}_i' - \dot{\mathbf{b}}'_i). \tag{35}
\]

Also, by dot multiplication both sides of Eq. (34) by \( \mathbf{s}_i' \), the following expression for \( \ddot{l}_i' \) is obtained

\[
\ddot{l}_i' = \mathbf{s}_i' \cdot (\dot{\mathbf{a}}_i' - \dot{\mathbf{b}}'_i). \tag{36}
\]

Substituting the obtained relations for \( \dot{\mathbf{a}}_i' \) from Eq. (24) and \( \dot{\mathbf{b}}'_i \) from Eq. (31), the above equation can be rewritten in the following compact form.

\[
\ddot{l}_i' = \mathbf{s}_i'[1_{3 \times 3} - \ddot{\mathbf{u}}_\mathbf{R}_1] \ddot{\mathbf{p}} - \mathbf{s}_i'[\mathbf{h}'(\gamma_i) \ddot{l}_i]. \tag{37}
\]

Now, relations between the velocities of the actuators and the moving plate are obtained. If the output vector, i.e.; the moving platform position denoted by \( \mathbf{x} = [\mathbf{p} \quad \Theta]^T \) and the input vector, i.e.; actuated joints is shown by \( \mathbf{q} = [\mathbf{q}_1 \quad \mathbf{q}_2]^T \) where

\[
\mathbf{q}_1 = \begin{bmatrix} l_1^1 & l_2^1 & \ldots & l_3^1 \\ l_1^2 & l_2^2 & \ldots & l_3^2 \end{bmatrix}^T, \tag{38}
\]


\[ \mathbf{q}_2 = \begin{bmatrix} \mathbf{l}_1 & \mathbf{l}_2 & \mathbf{l}_3 \end{bmatrix}^T. \]  

Eq. (37) can be written in the following matrix form

\[ \mathbf{J}_\mathbf{x} - \mathbf{J}_\mathbf{q} \dot{\mathbf{q}} = \mathbf{0}. \]  

where, \( \mathbf{J}_\mathbf{x} \) and \( \mathbf{J}_\mathbf{q} \) are \( 6 \times 6 \) and \( 6 \times 9 \) Jacobian matrices, namely,

\[
\mathbf{J}_\mathbf{q} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & e^1_l & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & e^2_l & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & e^3_l & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & e^4_l & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & e^5_l \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\end{bmatrix},
\]

\[
\mathbf{J}_\mathbf{x} = \begin{bmatrix}
\dot{\mathbf{J}}^1_l \\
\dot{\mathbf{J}}^2_l \\
\dot{\mathbf{J}}^3_l \\
\dot{\mathbf{J}}^4_l \\
\dot{\mathbf{J}}^5_l \\
\end{bmatrix},
\]

where

\[ e^i_l = s^T_i \mathbf{h}(\gamma_i). \]

and

\[
\dot{\mathbf{J}}^i_l = s^T_i [\mathbf{I}_{3 \times 3} - \mathbf{\tilde{u}}^l_{11}].
\]

The elements of the direct Jacobian matrix are dimensionally non-homogeneous. To obtain a dimensionally homogenous Jacobian matrix, the elements of the fourth, fifth and sixth columns of the direct Jacobian matrix are divided by characteristic length, \( l = (a + b)/2 \) [31,32].

Time derivative of Eq. (36) gives the acceleration of the lateral prismatic actuators as

\[
\ddot{\mathbf{p}}_l = s^T_l \cdot (\dot{\mathbf{a}}^l_l - \mathbf{\dot{b}}^l_l) + (\Omega^l_l \times s^l_l) \cdot (\dot{\mathbf{a}}^l_l - \mathbf{\dot{b}}^l_l),
\]

which can be expressed in the following form

\[
\ddot{\mathbf{p}}_l = s^T_l [\mathbf{I}_{3 \times 3} - \mathbf{u}^l_{11}] \left[ \frac{\mathbf{\ddot{p}}}{\Theta} \right] + s^T_l \mathbf{\dot{v}}^l_v - s^T_l \mathbf{h}(\gamma_l) \ddot{\mathbf{p}}_l - (\Omega^l_l \times s^l_l)^T \mathbf{h}^l_l = \mathbf{0},
\]

where

\[
\delta l^i_l = \sum_{i=1}^{3} f^i_l s^T_l [\mathbf{I}_{3 \times 3} - \mathbf{\tilde{u}}^l_{11}] \left[ \frac{\mathbf{\delta p}}{\Theta} \right] - s^T_l \mathbf{h}^l_l (\gamma^l_l) \delta l_i.
\]

6. Dynamic equations

Based on the kinematic analyses discussed in the previous section, a general formulation for the dynamic analyses of the proposed kinematically redundant mechanism can be presented using D’Alembert’s principle. For this purpose, the flexibility of the limbs and the friction and backlash in the joints are neglected.

Let \( f^i_l \) and \( f^i_l \) be the forces in the lateral and the base prismatic actuators, respectively. Therefore, for the present mechanism we have

\[
m_p \ddot{\mathbf{p}} + \sum_{i=1}^{3} m_b \ddot{\mathbf{l}}_i + (\mathbf{J} \omega + \omega \times \mathbf{J} \omega) \ddot{\omega} + \mathbf{m}_p \mathbf{g} \cdot \mathbf{p} + \sum_{i=1}^{3} f^i_l \ddot{\mathbf{l}}_i + \sum_{i=1}^{3} f^i_l \delta l^i_i = \mathbf{0}.
\]

Substituting, Eq. (47), into Eq. (46), we have

\[
m_p \ddot{\mathbf{p}} + \sum_{i=1}^{3} m_b \delta l_i + (\mathbf{J} \omega + \omega \times \mathbf{J} \omega) \delta \omega + \mathbf{m}_p \mathbf{g} \cdot \mathbf{p} + \sum_{i=1}^{3} f^i_l \delta l_i \\
+ \sum_{i=1}^{3} \sum_{j=1}^{2} f^i_l s^T_l [\mathbf{I}_{3 \times 3} - \mathbf{\tilde{u}}^l_{11}] \left[ \frac{\mathbf{\delta p}}{\Theta} \right] - \sum_{i=1}^{3} \sum_{j=1}^{2} f^i_l s^T_l \mathbf{h}^l_l (\gamma^l_l) \delta l_i = \mathbf{0}.
\]
Therefore, the equations of motion are obtained in the following forms

\[
\begin{bmatrix}
M_1 & 0 \\
0 & M_2
\end{bmatrix}
\begin{bmatrix}
\dot{x} \\
\dot{q}_2
\end{bmatrix}
+ \begin{bmatrix}
C & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\ddot{x} \\
\dot{q}_2
\end{bmatrix}
+ \begin{bmatrix}
D & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
f_1 \\
f_2
\end{bmatrix}
= \begin{bmatrix}
-P_1 & 0 \\
0 & -I_{3\times3}
\end{bmatrix}
\begin{bmatrix}
f_1 \\
f_2
\end{bmatrix}
\tag{49}
\]

where, matrices \( M_1, C, D \) and \( P_1 \) are

\[
M_1 = \begin{bmatrix}
m_b I_{3\times3} & 0_{3\times3} \\
0_{3\times3} & R_1^T[R_2]RR_1
\end{bmatrix}
\tag{50}
\]

\[
C = \begin{bmatrix}
0_{3\times3} \\
0_{3\times3}
\end{bmatrix}
\begin{bmatrix}
R_1^T \dot{\omega} R_2^T[RR_1^T] + R_1^T[R_2]RR_2
\end{bmatrix}
\tag{51}
\]

\[
D = \begin{bmatrix}
m_p g \\
0_{3\times1}
\end{bmatrix}
\tag{52}
\]

\[
P_1 = \begin{bmatrix}
-s_1^1 s_1^T & -s_1^2 s_2^T & \cdots & -s_1^3 s_3^T \\
-s_2^1 u_1^T & s_2^2 u_2^T & \cdots & s_2^3 u_3^T \\
0 & 0 & \cdots & 0
\end{bmatrix}
\tag{53}
\]

and matrices \( M_2 \) and \( P_2 \) are as

\[
M_2 = \begin{bmatrix}
m_b & 0 & 0 \\
0 & m_b & 0 \\
0 & 0 & m_b
\end{bmatrix}
\tag{54}
\]

\[
P_2 = \begin{bmatrix}
e_1^1 & e_1^2 & 0 & 0 & 0 & 0 \\
e_2^1 & e_2^2 & 0 & 0 & 0 & 0 \\
e_3^1 & e_3^2 & 0 & 0 & 0 & 0
\end{bmatrix}
\tag{55}
\]

while, the actuator force vectors \( f_1 \) and \( f_2 \) are

\[
f_1 = \begin{bmatrix}
f_1^1 \\
f_1^2 \\
f_1^3
\end{bmatrix}, \quad f_2 = \begin{bmatrix}
f_2^1 \\
f_2^2 \\
f_2^3
\end{bmatrix}.
\tag{56}
\]

7. Control strategy

In what follows, inverse dynamic control method with a non-linear disturbance observer is discussed for the control of kinematically redundant parallel mechanisms.

7.1. Inverse dynamic control

The dynamic equations of the considered class of kinematically redundant parallel mechanism can be written as

\[
M \ddot{r} + C(r, \dot{r}) \dot{r} + G(r) = Pu + d.
\tag{57}
\]

where, the \( m \) task space degrees of freedom, \( x \) and the \( (n-m) \) of the redundant joint variables, \( q_2 \) are selected as the generalized coordinates, \( r = [x^T \ q_2^T]^T \). In this formulation, \( M \) is the mass matrix, \( C(r, \dot{r}) \) denotes the coriolis and centrifugal matrix, \( G(r) \) and \( d \) are the gravity vector and disturbance vector due to interactions with external environment, respectively, \( u \) denotes the actuator forces and \( P \) is the coefficient matrix of the actuator forces.

Given the desired motion variables in the task space, \( x_d \), the desired motion variables in the joint space, \( q_d \), are obtained using redundancy resolution algorithm, and the error vector \( e = r_d - r \) is used for motion control of the redundant mechanism. The general configuration of the inverse dynamic control is depicted in Fig. (4).

The controller output wrench applied to the parallel mechanism is given as \[25\]

\[
u = P^{-1}(Ma + f_d - d).
\tag{58}
\]

in which

\[
f_d = C(r, \dot{r}) \dot{r} + G(r).
\tag{59}
\]

\[
a = \dot{x}_d + K_p \dot{e} + K_v e.
\tag{60}
\]

where, \( K_p \) and \( K_v \) are the proportional and derivative control gain matrices and \( \dot{d} \) is the estimated disturbance. Using Eqs. (57) and (58), the closed-loop dynamic formulation of the mechanism can be written as

\[
M(\ddot{e} + K_p \dot{e} + K_v e) = d - \dot{d}.
\tag{61}
\]

By choosing appropriate gains for the controller, the error equation is exponentially stable \[25\].
7.2. Disturbance observer

Considering slowly varying disturbance ($\dot{\mathbf{d}} \approx 0$), the disturbance observer can be suggested given as \cite{33,34}

$$\hat{\mathbf{d}} = \mathbf{L}(\mathbf{r}, \mathbf{\dot{r}})(\mathbf{d} - \hat{\mathbf{d}}).$$

(62)

where, $\mathbf{L}(\mathbf{r}, \mathbf{\dot{r}})$ is the observer matrix.

Substituting $\mathbf{d}$ from equations of motions, the above equations can be solved to estimate the disturbance. Since, the accelerations of states are not available; this method can't be implemented to estimate the disturbance. Thus, the auxiliary variable, $\mathbf{z}$ is defined as

$$\mathbf{z} = \hat{\mathbf{d}} - \mathbf{p}(\mathbf{r}, \mathbf{\dot{r}}).$$

(63)

in which, $\mathbf{p}(\mathbf{r}, \mathbf{\dot{r}})$ is the observer vector. In order to obtain a simple relation for the observer vector, the following relation for the observer gain matrix is selected

$$\mathbf{L}(\mathbf{r}, \mathbf{\dot{r}}) = \alpha \mathbf{M}^{-1}.$$  

(64)
in which, the parameter $\alpha$ is a positive constant. Considering the following relation between the observer vector and the derivative of observer matrix

$$L(r, \dot{r})\ddot{\tilde{r}} = \frac{d\mathbf{p}(r, \dot{r})}{dt}.$$  \hspace{1cm} (65)
the following relation for the observer vector can be obtained
\[ \mathbf{p}(\mathbf{r}, \dot{\mathbf{r}}) = \alpha \dot{\mathbf{r}}. \] (66)

Subsequently, using the assumed relations for the observer vector and observer gain matrix, a relation for the auxiliary variable is obtained. Time derivative of Eq. (63), and using Eqs. (62) and (65), yields
\[ \dot{\mathbf{z}} = \mathbf{L}(\mathbf{r}, \dot{\mathbf{r}})(\dot{\mathbf{d}} - \dot{\mathbf{d}}) + \mathbf{L}(\mathbf{r}, \dot{\mathbf{r}})\mathbf{M} \dot{\mathbf{r}}. \] (67)
Substituting \( \mathbf{d} \) from the equations of motion, the above differential equations can be solved to estimate the disturbance.
\[ \dot{\mathbf{z}} = -\mathbf{L}(\mathbf{r}, \dot{\mathbf{r}})\mathbf{z} + \mathbf{L}(\mathbf{r}, \dot{\mathbf{r}})(\mathbf{C}(\mathbf{r}, \dot{\mathbf{r}})\dot{\mathbf{r}} + \mathbf{G}(\mathbf{r}) - \mathbf{Pu} - \mathbf{p}(\mathbf{r}, \dot{\mathbf{r}})). \] (68)
In sequence, the stability of the designed observer is investigated. Time derivative of the observer error yields
\[ \dot{\mathbf{e}} = \dot{\mathbf{d}} - \dot{\mathbf{d}} = -\mathbf{L}(\mathbf{r}, \dot{\mathbf{r}})\mathbf{e}. \] (69)
This equation is marginally stable, if the observer matrix is positive definite [25].

8. Simulations and results

In this section, the proposed redundancy resolution method is implemented for the control of the proposed spatial mechanism. For this purpose, the considered geometric parameters are \( a = 0.1 \) m, \( b = 0.1 \) m and \( h = 0.05 \) m. The mass of the end-effector and the lumped masses are considered to be \( m_p = 1.5 \) kg and \( m_b = 0.2 \) kg, respectively. Also, the non-zero components of inertia tensor are assumed as \( J_{xx} = J_{yy} = 0.02 \) kg.m² and \( J_{zz} = 0.04 \) kg.m². The control gain matrices are selected \( \mathbf{K}_p = \mathbf{K}_d = 40I_{6 \times 6} \) and the weighting coefficients are \( \lambda_i = 0.1 \) for \( i = 1, \ldots, 6 \).

In the presented simulations, it is assumed that the orientation of the end-effector is constant and point P tracks a circular path with radius \( r = 0.2 \) m as
\[ \mathbf{x} = \begin{bmatrix} 0.2 \sin(\pi t) & 0.2 \cos(\pi t) & 0.5 & 0 & 0 & 0 \end{bmatrix}^T \] (70)
The components of the cutting force are given as
\[ f_x = -p_t \sin(\pi t) + p_r \cos(\pi t) \]
\[ f_y = p_t \cos(\pi t) + p_r \sin(\pi t) \]
\[ f_z = p_z \]
(71)
in which, \( p_t, p_r \) and \( p_z \) are tangential, radial and axial forces, respectively and are considered to be \( p_t = p_r = p_z = 1 \).
Fig. 9. Forces in the six lateral prismatic actuators of the redundant mechanism.

Fig. 10. Forces in the six lateral prismatic actuators of the non-redundant mechanism.
Using the proposed redundancy resolution, the lengths of three base prismatic actuators versus time are plotted in Fig. (5). It should be mentioned that, the stroke limitation is specified by $\ell_{\text{min}} = 0.4 \text{ m}$ and $\ell_{\text{max}} = 0.8 \text{ m}$.

The optimization problem in each time interval is solved using FMINCON, included in MATLAB’s optimization toolbox, finding the minimum of a constrained nonlinear multivariable function [35]. The default algorithm interior-point is selected for optimization that can handle large, sparse problems and satisfies bounds at all iterations.

In the presented simulations, the time intervals are equal and considered $\Delta t = 1.0 \text{ s}$. The run time for solving optimization problem is less than 0.3 s for all of the interval times. Thus, the aforementioned limitation of the proposed strategy is not problematic.

To investigate the ability of the proposed control strategy, the task space trajectory traced by the moving plate in the presence of disturbance is depicted in Fig. (6). The capability of the controller with the disturbance observer in eliminating the disturbances is illustrated. The estimated disturbance is depicted in Fig. (7).

To investigate the performance of the proposed mechanism, the condition number of the redundant mechanism in the given trajectory is compared with the similar size non-redundant mechanism, see Fig. (8). The geometric parameters of the moving plate of the non-redundant mechanism are considered similar with those of the redundant actuators. Three base prismatic actuators are fixed with the same lengths $l_i = 0.5 \text{ m}$ $i = 1, 2, 3$.

The actuator forces for tracking the given path are obtained and compared with those of a similar non-redundant mechanism. The forces in the six lateral prismatic actuators in the redundant and non-redundant mechanism are plotted in Figs. (9) and (10), respectively. Also, the actuator forces in the three prismatic actuators on the base of the redundant mechanism are plotted in Fig. (11).

9. Conclusion

This paper addressed the issue of redundancy resolution and control of a novel kinematically redundant parallel mechanism. An online motion planning algorithm was introduced for singularly avoidance while considering the full dynamics of the mechanism. The method is general and can be implemented real time for actuation of kinematically redundant parallel mechanisms.

One of the potential applications of the proposed kinematically redundant mechanism is as a milling machine tool with high dexterity and accuracy. The dynamic equations of the proposed mechanism were derived and the inverse dynamic control strategy with nonlinear disturbance observer was designed. The numerical simulations illustrated the advantages of the proposed redundant mechanism over the non-redundant one with similar structure.
References